Sharp two–sided heat kernel estimates for critical Schrödinger operators on bounded domains

Stathis Filippas[1,4] & Luisa Moschini[2] & Achilles Tertikas[3,4]

Department of Applied Mathematics[1]
University of Crete, 71409 Heraklion, Greece
filippas@tem.uoc.gr

Dipartimento di Metodi e Modelli Matematici per le Scienze Applicate[2],
University of Rome "La Sapienza", 00161 Rome, Italy
moschini@dmmm.uniroma1.it

Department of Mathematics[3]
University of Crete, 71409 Heraklion, Greece
tertikas@math.uoc.gr

Institute of Applied and Computational Mathematics[4],
FORTH, 71110 Heraklion, Greece

Abstract

On a smooth bounded domain $\Omega \subset \mathbf{R}^N$ we consider the Schrödinger operators $-\Delta - V$, with V being either the critical borderline potential $V(x) = (N-2)^2/4 \; |x|^{-2}$ or $V(x) = (1/4) \; \mathrm{dist}(x, \partial\Omega)^{-2}$, under Dirichlet boundary conditions. In this work we obtain sharp two-sided estimates on the corresponding heat kernels. To this end we transform the Schrödinger operators into suitable degenerate operators, for which we prove a new parabolic Harnack inequality up to the boundary. To derive the Harnack inequality we have established a series of new inequalities such as improved Hardy, logarithmic Hardy Sobolev, Hardy-Moser and weighted Poincaré. As a byproduct of our technique we are able to answer positively to a conjecture of E. B. Davies.

AMS Subject Classification: 35K65, 26D10 (35K20, 35B05.)
Keywords: Singular heat equation, heat kernel estimates, logarithmic Sobolev inequalities, Hardy inequality, Hardy-Moser inequality, parabolic Harnack inequality, degenerate elliptic operators.

1 Introduction and main results

Harnack inequalities have been extremely useful in the study of solutions of elliptic and parabolic equations, starting from the pioneering works of De Giorgi [DG], Nash [N] and Moser [Mo1], [Mo2]. They are used to prove Hölder continuity of solutions, strong maximum principles, Liouville properties, as well as sharp two-sided heat kernel estimates. In particular, we should mention the influential works of Aronson [A] and Li and Yau [LY] where heat kernel estimates were obtained via parabolic Harnack inequalities.

In fact, in certain cases, the parabolic Harnack inequality is equivalent to sharp two–sided heat kernel estimates. This is the case when dealing with second order uniformly elliptic operators in divergence form on $\mathbf{R}^N$, or more generally with weighted Laplacians on complete Riemannian manifolds; see the works of Fabes and Stroock [FS], Grigoryan [G1], and Saloff–Coste [SC1]. This equivalence has been also used in order to get sharp two–sided estimates for Schrödinger operators in $\mathbf{R}^N$. For instance, the case of a potential that is regular and decays like $|x|^{-2}$ at infinity was studied by Davies and Simon [DS2], where pointwise upper bounds for the heat kernel were derived. The picture was later completed by Grigoryan [G2] where sharp two sided estimates were provided by means of a parabolic Harnack inequality. A recent survey on heat kernels on weighted manifolds can also be found in [G2].

As it was shown in the works of Fabes, Kenig and Serapioni [FKS], and Chiarenza and Serapioni [CS], parabolic Harnack inequalities follow after establishing Poincaré and Sobolev inequalities as well as a doubling volume growth condition. Moreover, on complete Riemannian manifolds parabolic Harnack inequalities are equivalent to Poincaré inequality and a doubling volume growth condition as explained by Grigoryan and Saloff–Coste in [GSC], [SC2].

Since the work of Baras and Goldstein [BG], the existence or nonexistence of solutions to the partial differential equation

$$u_t = \Delta u + V u, \tag{1.1}$$

with a potential V involving the inverse square of the distance function have been widely investigated. See [BG], Brezis and Vázquez [BV], Cabré and Martel [CM], as well as Vázquez and Zuazua [VZ], for the case $V(x) = c|x|^{-2}$ and [CM] for the case $V(x) = cd^{-2}(x)$ on a bounded domain Ω, where $d(x) = \operatorname{dist}(x, \partial\Omega)$.

Concerning the case where $V(x) = c|x|^{-2}$ with $c < (N-2)^2/4$, sharp two–sided heat kernel estimates have been obtained in $\mathbf{R}^N$, see [MT1], [MT2] where the approach of [GSC] on complete Riemannian manifolds has been used, after a suitable transformation; see also [MS] for a different method.

On the other hand few results are known in the case of incomplete Riemannian manifolds, as it is for example the case of bounded domains in $\mathbf{R}^N$. To our knowledge the only sharp two sided estimates in this case, concern the standard Dirichlet Laplacian on a smooth bounded domain $\Omega \subset \mathbf{R}^N$, first studied by Davies and Simon in [D1], [D2], [DS1], and recently completed by Zhang [Z]. We note that in the case of a bounded domain, the asymptotic of the heat kernel is different for small time than it is for large time. In fact, for the heat kernel $h_D(t,x,y)$ of the standard Dirichlet Laplacian and for two positive constants $C_1 \le C_2$, we have for small time

$$C_1 \min\left\{1, \frac{d(x)d(y)}{t}\right\} t^{-\frac{N}{2}} e^{-C_2 \frac{|x-y|^2}{t}} \le h_D(t,x,y) \le C_2 \min\left\{1, \frac{d(x)d(y)}{t}\right\} t^{-\frac{N}{2}} e^{-C_1 \frac{|x-y|^2}{t}}, \tag{1.2}$$

whereas for large time

$$C_1\ d(x)\ d(y)\ e^{-\lambda_1 t} \le h_D(t,x,y) \le C_2\ d(x)\ d(y)\ e^{-\lambda_1 t}, \tag{1.3}$$

for all $x, y \in \Omega$; here λ_1 is the first Dirichlet eigenvalue.

In this work, our main interest is in obtaining sharp two–sided estimates for the heat kernel of the Schrödinger operator $-\Delta - V$ under Dirichlet boundary conditions, on a smooth bounded domain $\Omega \subset \mathbf{R}^N$ for the following critical borderline potentials: $V(x) = ((N-2)^2/4)|x|^{-2}$ or $V(x) = (1/4)d^{-2}(x)$.

Throughout this work Ω is a C^2 bounded domain of $\mathbf{R}^N$ containing the origin and $d(x) = \operatorname{dist}(x, \partial\Omega)$. We first consider, for $N \ge 3$, the case $V(x) = ((N-2)^2/4)|x|^{-2}$, $x \in \Omega$ and we formally define the operator K by $Ku = -\Delta u - \frac{(N-2)^2}{4|x|^2} u$, $\quad u|_{\partial\Omega} = 0$. More precisely, the Schrödinger operator K is defined in $L^2(\Omega)$ as the generator of the symmetric form

$$\mathcal{K}[u_1, u_2] := \int_\Omega \left(\nabla u_1 \nabla u_2 - \frac{(N-2)^2}{4|x|^2} u_1 u_2 \right) dx \ ,$$

namely, if

$$D(K) := \left\{ u \in H(\Omega) : -\Delta u - \frac{(N-2)^2}{4|x|^2} u \in L^2(\Omega) \right\},$$

$$Ku := -\Delta u - \frac{(N-2)^2}{4|x|^2}u \ \text{ for any } u \in D(K) \ , \tag{1.4}$$

where $H(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ in the norm

$$u \to ||u||_{H(\Omega)} := \left\{ \int_\Omega \left(|\nabla u|^2 - \frac{(N-2)^2}{4|x|^2}u^2 \right) dx \right\}^{\frac{1}{2}} \ . \tag{1.5}$$

Let us recall that $H(\Omega) \subset W_0^{1,q}(\Omega)$ for any $1 \leq q < 2$, due to the results in Subsection 4.1 of [VZ].

It follows, using Hardy inequality, that K is a nonnegative self-adjoint operator on $L^2(\Omega)$ such that for every $t > 0$, e^{-Kt} has an integral kernel, that is, $e^{-Kt}u_0(x) := \int_\Omega k(t,x,y)u_0(y)dy$ where $k(t,x,y)$ is the heat kernel of K. The first Dirichlet eigenvalue of K can be defined by

$$\lambda_1 := \inf_{0 \neq \varphi \in C_0^\infty(\Omega)} \frac{\int_\Omega \left(|\nabla \varphi|^2 - \frac{(N-2)^2}{4|x|^2}\varphi^2 \right) dx}{\int_\Omega \varphi^2 dx} \ , \tag{1.6}$$

with $\lambda_1 > 0$, due to [BV]. Moreover there exists a positive function $\varphi_1 \in H(\Omega)$ satisfying

$$-\Delta \varphi_1 - \frac{(N-2)^2}{4|x|^2}\varphi_1 = \lambda_1 \varphi_1, \quad \text{in} \ \ \Omega, \quad \varphi_1 = 0, \ \ \text{on} \ \ \partial\Omega,$$

see for example Davila and Dupaigne [DD].

We then have the following sharp two-sided heat kernel estimate on K for small time

Theorem 1.1 *Let $\Omega \subset \mathbf{R}^N$, $N \geq 3$, be a smooth bounded domain containing the origin. Then there exist positive constants C_1, C_2, with $C_1 \leq C_2$, and $T > 0$ depending on Ω such that*

$$C_1 \min\left\{ (|x| + \sqrt{t})^{\frac{N-2}{2}} (|y| + \sqrt{t})^{\frac{N-2}{2}}, \frac{d(x)d(y)}{t} \right\} (|x||y|)^{\frac{2-N}{2}} t^{-\frac{N}{2}} e^{-C_2 \frac{|x-y|^2}{t}} \leq$$

$$\leq k(t,x,y) \leq C_2 \min\left\{ (|x| + \sqrt{t})^{\frac{N-2}{2}} (|y| + \sqrt{t})^{\frac{N-2}{2}}, \frac{d(x)d(y)}{t} \right\} (|x||y|)^{\frac{2-N}{2}} t^{-\frac{N}{2}} e^{-C_1 \frac{|x-y|^2}{t}},$$

for all $x, y \in \Omega$ and $0 < t \leq T$.

Concerning the large time asymptotic we have:

Theorem 1.2 *Let $\Omega \subset \mathbf{R}^N$, $N \geq 3$, be a smooth bounded domain containing the origin. Then there exist two positive constants C_1, C_2, with $C_1 \leq C_2$, such that*

$$C_1 \ d(x) \ d(y) \ (|x||y|)^{\frac{2-N}{2}} e^{-\lambda_1 t} \leq k(t,x,y) \leq C_2 \ d(x) \ d(y) \ (|x||y|)^{\frac{2-N}{2}} e^{-\lambda_1 t},$$

for all $x, y \in \Omega$ and $t > 0$ large enough; here λ_1 is defined in (1.6).

To prove the above Theorem 1.2 we have shown a new improved Hardy inequality which is of independent interest; see Theorem 3.2.

We next consider the case where the Schrödinger operator H has a potential with critical borderline singularity at the boundary $Hu = -\Delta u - \frac{1}{4d^2(x)}u \ , \quad u|_{\partial\Omega} = 0$; here $N \geq 2$ and Ω is a convex domain. More precisely, the Schrödinger operator H is defined in $L^2(\Omega)$ as the generator of the symmetric form

$$\mathcal{H}[u_1, u_2] := \int_\Omega \left(\nabla u_1 \nabla u_2 - \frac{1}{4d^2(x)} u_1 u_2 \right) dx \ ,$$

namely if

$$D(H) := \left\{ u \in W(\Omega) : -\Delta u - \frac{1}{4d^2(x)} u \in L^2(\Omega) \right\},$$

$$Hu := -\Delta u - \frac{1}{4d^2(x)} u \ \text{ for any } u \in D(H) \ , \tag{1.7}$$

where $W(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ in the norm

$$u \to ||u||_{W(\Omega)} := \left\{ \int_\Omega \left(|\nabla u|^2 - \frac{1}{4d^2(x)} u^2 \right) dx \right\}^{\frac{1}{2}} \ .$$

Let us recall that $W(\Omega) \subset W_0^{1,q}(\Omega)$ for any $1 \leq q < 2$, due to Theorem B in [BFT1].

Then, due to Hardy inequality, H is a nonnegative self-adjoint operator on $L^2(\Omega)$ such that for every $t > 0$, e^{-Ht} has an integral kernel, that is, $e^{-Ht} u_0(x) := \int_\Omega h(t,x,y) u_0(y) dy$; here $h(t,x,y)$ denotes the heat kernel of H. The first Dirichlet eigenvalue of H is defined by

$$\lambda_1 := \inf_{0 \neq \varphi \in C_0^\infty(\Omega)} \frac{\int_\Omega \left(|\nabla \varphi|^2 - \frac{1}{4d^2(x)} \varphi^2 \right) dx}{\int_\Omega \varphi^2 dx} \ . \tag{1.8}$$

It is known that $\lambda_1 > -\infty$ for any bounded domain Ω, and $\lambda_1 > 0$ if Ω is convex, see [BM]. Moreover there exists a positive function $\varphi_1 \in W(\Omega)$ satisfying

$$-\Delta \varphi_1 - \frac{1}{4d^2(x)} \varphi_1 = \lambda_1 \varphi_1, \quad \text{in } \ \Omega, \quad \varphi_1 = 0, \quad \text{on } \quad \partial\Omega;$$

see for example [DD].

We then have the following sharp two-sided heat kernel estimate on H for small time

Theorem 1.3 *Let $\Omega \subset \mathbf{R}^N$, $N \geq 2$, be a smooth bounded and convex domain. Then there exist positive constants C_1, C_2, with $C_1 \leq C_2$, and $T > 0$ depending on Ω such that*

$$C_1 \min\left\{ 1, \frac{d^{\frac{1}{2}}(x) d^{\frac{1}{2}}(y)}{t^{\frac{1}{2}}} \right\} t^{-\frac{N}{2}} e^{-C_2 \frac{|x-y|^2}{t}} \leq h(t,x,y) \leq C_2 \min\left\{ 1, \frac{d^{\frac{1}{2}}(x) d^{\frac{1}{2}}(y)}{t^{\frac{1}{2}}} \right\} t^{-\frac{N}{2}} e^{-C_1 \frac{|x-y|^2}{t}},$$

for all x, $y \in \Omega$ and $0 < t \leq T$.

We next complement this with the large time behavior:

Theorem 1.4 *Let $\Omega \subset \mathbf{R}^N$, $N \geq 2$, be a smooth bounded and convex domain. Then there exist two positive constants C_1, C_2, with $C_1 \leq C_2$, such that*

$$C_1 \ d^{\frac{1}{2}}(x) \ d^{\frac{1}{2}}(y) \ e^{-\lambda_1 t} \leq h(t,x,y) \leq C_2 \ d^{\frac{1}{2}}(x) \ d^{\frac{1}{2}}(y) \ e^{-\lambda_1 t},$$

for all x, $y \in \Omega$ and $t > 0$ large enough; here λ_1 is defined in (1.8).

The two-sided estimates in Theorems 1.1 and 1.3 are obtained as a consequence of a new parabolic Harnack inequality up to the boundary, for a suitable degenerate elliptic operator. Let us present a model operator in this direction. For this we consider classical solutions of

$$v_t = \frac{1}{d^\alpha(y)} div(d^\alpha(y) \nabla v), \tag{1.9}$$

(actually solutions are considered as weak solutions, for the precise formulation we refer to Definition 2.9 with $\lambda = 0$ there, note that due to elliptic regularity, any solution is smooth away from the boundary of Ω).

Then, the following Harnack inequality holds true:

Theorem 1.5 (Parabolic Harnack inequality up to the boundary). *Let $N \geq 2$, $\alpha \geq 1$ and $\Omega \subset \mathbf{R}^N$ be a smooth bounded domain. Then, there exist positive constants C_H and $R = R(\Omega)$ such that for $x \in \Omega$, $0 < r < R$ and for any positive solution $v(y,t)$ of (1.9) in $\{\mathcal{B}(x,r) \cap \Omega\} \times (0, r^2)$, the following estimate holds true*

$$ess\ sup_{(y,t)\in\left\{\mathcal{B}(x,\frac{r}{2})\cap\Omega\right\}\times(\frac{r^2}{4},\frac{r^2}{2})}\ \ v(y,t) \leq C_H\ ess\ inf_{(y,t)\in\left\{\mathcal{B}(x,\frac{r}{2})\cap\Omega\right\}\times(\frac{3}{4}r^2,r^2)}\ \ v(y,t)\ . \tag{1.10}$$

Here $\mathcal{B}(x,r)$ denotes roughly speaking an N dimensional cube centered at x and having size r (see Definition 2.1). The restriction on α in Theorem 1.5 is sharp, since in the *weakly degenerate* case, where $0 < \alpha < 1$, even the elliptic Harnack fails. Indeed, let $\Omega := B(0,1)$, then $v(y) := \int_{|y|}^{1} \frac{ds}{(1-s)^\alpha s^{N-1}}$ is a positive solution of $div(d^\alpha(y)\nabla v) = 0$ for $1/2 < |y| < 1$, with $v(1) = 0$. The natural analogue of Theorem 1.5 in the weakly degenerate case, that is $0 < \alpha < 1$, is a Harnack inequality for the ratio of any two positive solutions; in the elliptic case this is done in [FKJ] and by a probabilistic approach in [Ga].

To derive heat kernel estimates we define the operator $L := -\frac{1}{d^\alpha(x)} div(d^\alpha(x)\nabla)$ in $L^2(\Omega, d^\alpha(x)\ dx)$ as the generator of the symmetric form

$$\mathcal{L}[v_1, v_2] := \int_\Omega d^\alpha(x)\nabla v_1 \nabla v_2\ dx\ ,$$

namely

$$D(L) := \left\{ v \in H_0^1(\Omega, d^\alpha(x)\ dx) : -\frac{1}{d^\alpha(x)} div(d^\alpha(x)\nabla v) \in L^2(\Omega, d^\alpha(x)\ dx) \right\},$$

$$Lv := -\frac{1}{d^\alpha(x)} div(d^\alpha(x)\nabla v) \text{ for any } v \in D(L)\ , \tag{1.11}$$

where $H_0^1(\Omega, d^\alpha(x)\ dx)$ denotes the closure of $C_0^\infty(\Omega)$ in the norm

$$v \to ||v||_{H^1_\alpha} := \left\{ \int_\Omega d^\alpha(x) \left(|\nabla v|^2 + v^2\right)\ dx \right\}^{\frac{1}{2}}\ . \tag{1.12}$$

We should emphasize that for $\alpha \geq 1$, one has $H_0^1(\Omega, d^\alpha(x)\ dx) = H^1(\Omega, d^\alpha(x)\ dx)$, see Theorem 2.11.

Let us note that L is a nonnegative self-adjoint operator on $L^2(\Omega, d^\alpha(y)dy)$ such that for every $t > 0$, e^{-Lt} has a integral kernel, that is $e^{-Lt}v_0(x) := \int_\Omega l(t,x,y)v_0(y)d^\alpha(y)dy$; the existence of the heat kernel $l(t,x,y)$ can be proved arguing as in [DS1].

Then, arguing as in [GSC], and using the parabolic Harnack inequality up to the boundary (Theorem 1.5), we obtain the following sharp two-sided estimates for the heat kernel generated by L, for small time.

The estimate of Theorem 1.3 is a consequence of Theorem 1.5 and corresponds to the extreme value $\alpha = 1$. We refer to Theorem 2.10 for a more general result that leads to Theorem 1.1.

The existence of a uniform upper bound on the size of the admissible "balls" denoted by $R(\Omega)$, in Theorem 1.5 is necessary, because otherwise the nonexistence of an upper bound would imply two-sided heat kernel estimates that are the same for small time and large time, which is not the case at least for $\alpha = 1$, due to Theorems 1.3 and 1.4.

Theorem 1.6 *Let $\alpha \geq 1$, $N \geq 2$ and $\Omega \subset \mathbf{R}^N$ be a smooth bounded domain. Then there exist positive constants C_1, C_2, with $C_1 \leq C_2$, and $T > 0$ depending on Ω such that*

$$C_1 \min\left\{\frac{1}{t^{\frac{\alpha}{2}}}, \frac{1}{d^{\frac{\alpha}{2}}(x)d^{\frac{\alpha}{2}}(y)}\right\} t^{-\frac{N}{2}} e^{-C_2\frac{|x-y|^2}{t}} \leq l(t,x,y) \leq C_2 \min\left\{\frac{1}{t^{\frac{\alpha}{2}}}, \frac{1}{d^{\frac{\alpha}{2}}(x)d^{\frac{\alpha}{2}}(y)}\right\} t^{-\frac{N}{2}} e^{-C_1\frac{|x-y|^2}{t}},$$

for all $x, y \in \Omega$ and $0 < t \leq T$.

So far we have considered special potentials V. However as we shall see next we can obtain much more general results. For instance we consider the operator $E := -\Delta - V$ where the potential V is such that

$$V(x) = V_1(x) + V_2(x)\ , \tag{1.13}$$

where

$$|V_1(x)| \le \frac{1}{4d^2(x)}\ , \quad V_2(x) \in L^p(\Omega),\ p > \frac{N}{2}\ . \tag{1.14}$$

We also suppose that

$$\lambda_1 := \inf_{0\neq\varphi\in C_0^\infty(\Omega)} \frac{\int_\Omega \left(|\nabla\varphi|^2 - V\varphi^2\right)dx}{\int_\Omega \varphi^2 dx} > 0, \tag{1.15}$$

and that to λ_1 there corresponds a positive eigenfunction φ_1 satisfying for all $x \in \Omega$ the following estimate,

$$c_1 d^{\frac{\alpha}{2}}(x) \le \varphi_1(x) \le c_2 d^{\frac{\alpha}{2}}(x)\ , \quad \text{for some} \quad \alpha \ge 1, \tag{1.16}$$

and for $c_1,\ c_2$ two positive constants.

Then as before it can be shown that E is a well defined nonnegative self-adjoint operator on $L^2(\Omega)$ such that for every $t > 0$, e^{-Et} has a integral kernel, that is $e^{-Et}u_0(x) := \int_\Omega e(t,x,y)u_0(y)dy$. We consider positive solutions of

$$u_t = -Eu\ ; \tag{1.17}$$

then our first result reads

Corollary 1.7 *For $N \ge 2$, let $\Omega \subset \mathbf{R}^N$ be a smooth bounded domain. Suppose that (1.13), (1.14), (1.15) and (1.16) are satisfied. Then, there exist positive constants C_H and $R = R(\Omega)$ such that for $x \in \Omega$, $0 < r < R$ and for any positive solution $u(y,t)$ of (1.17) in $\{\mathcal{B}(x,r)\cap\Omega\} \times (0,r^2)$ we have the estimate*

$$\textit{ess sup}_{(y,t)\in\{\mathcal{B}(x,\frac{r}{2})\cap\Omega\}\times(\frac{r^2}{4},\frac{r^2}{2})}\ u(y,t)d^{-\frac{\alpha}{2}}(y) \le C_H\ \textit{ess inf}_{(y,t)\in\{\mathcal{B}(x,\frac{r}{2})\cap\Omega\}\times(\frac{3}{4}r^2,r^2)}\ u(y,t)d^{-\frac{\alpha}{2}}(y)\ .$$

Our result in the case $\alpha = 2$ is basically the local comparison principle by Fabes, Garofalo and Salsa [FGS] in the case of Schrödinger operator (see Remark 2.16, that covers the uniformly elliptic case).

As we have seen before the parabolic Harnack inequality yields sharp two-sided estimates for the heat kernel $e(t,x,y)$, corresponding to the operator E. In particular we have:

Corollary 1.8 *For $N \ge 2$, let $\Omega \subset \mathbf{R}^N$ be a smooth bounded domain. Suppose that (1.13), (1.14), (1.15) and (1.16) are satisfied. Then there exist positive constants C_1, C_2, with $C_1 \le C_2$, and $T > 0$ depending on Ω such that for any $x, y \in \Omega$ and $0 < t \le T$*

$$C_1 \min\left\{1, \frac{d^{\frac{\alpha}{2}}(x)d^{\frac{\alpha}{2}}(y)}{t^{\frac{\alpha}{2}}}\right\} t^{-\frac{N}{2}} e^{-C_2\frac{|x-y|^2}{t}} \le e(t,x,y) \le C_2 \min\left\{1, \frac{d^{\frac{\alpha}{2}}(x)d^{\frac{\alpha}{2}}(y)}{t^{\frac{\alpha}{2}}}\right\} t^{-\frac{N}{2}} e^{-C_1\frac{|x-y|^2}{t}},$$

whereas for $t > T$ we have

$$C_1\ d^{\frac{\alpha}{2}}(x)d^{\frac{\alpha}{2}}(y)e^{-\lambda_1 t} \le e(t,x,y) \le C_2\ d^{\frac{\alpha}{2}}(x)d^{\frac{\alpha}{2}}(y)e^{-\lambda_1 t}\ .$$

As a byproduct of our method we can answer a conjecture by E. B. Davies (Conjecture 7 in [D2]) in the case of the Schrödinger operator (see Section 4.4 for a more general case). For this let us introduce the Green function associated to E, that is

$$G_E(x,y) = \int_0^\infty e(t,x,y)dt\ , \tag{1.18}$$

then we have

Corollary 1.9 *For $N \geq 3$, let $\Omega \subset \mathbf{R}^N$ be a smooth bounded domain. Suppose that (1.13), (1.14), (1.15) and (1.16) are satisfied. Then there exist two positive constants C_1, C_2, with $C_1 \leq C_2$, such that for any $x, y \in \Omega$*

$$C_1 \min\left\{\frac{1}{|x-y|^{N-2}}, \frac{d^{\frac{\alpha}{2}}(x)d^{\frac{\alpha}{2}}(y)}{|x-y|^{N+\alpha-2}}\right\} \leq G_E(x,y) \leq C_2 \min\left\{\frac{1}{|x-y|^{N-2}}, \frac{d^{\frac{\alpha}{2}}(x)d^{\frac{\alpha}{2}}(y)}{|x-y|^{N+\alpha-2}}\right\}.$$

Davies conjecture corresponds to our result in the case $\alpha = 2$, we should note however that other values of $\alpha \geq 1$ are possible.

The structure of the paper is as follows. In Section 2 we prove the new parabolic Harnack inequality up to the boundary for a doubly degenerate elliptic operator, as well as, the two sided small time heat kernel estimates that can be deduced from it. In Section 3 we present the proof of the above mentioned results concerning the Schrödinger potential having critical singularity at the origin, while Section 4 treats the case of the Schrödinger operator having critical singularity on the boundary.

Acknowledgment This work was largely done whilst the second author was visiting the University of Crete and FORTH in Heraklion, the hospitality of which is acknowledged. This research has been partially supported by the RTN European network Fronts–Singularities, HPRN-CT-2002-00274.

2 Parabolic Harnack inequality up to the boundary for degenerate operators

In this section we prove a new parabolic Harnack inequality up to the boundary for the doubly degenerate elliptic operator in divergence form

$$L_\alpha^\lambda := -\frac{1}{|x|^\lambda d^\alpha(x)} div(|x|^\lambda d^\alpha(x)\nabla), \tag{2.1}$$

for any $\alpha \geq 1$ and $\lambda \in [2-N, 0]$. Our approach is to first obtain a doubling volume-growth condition (see Corollary 2.4), then a local weighted Poincaré inequality (see Theorem 2.5), and finally a local weighted Moser inequality. In fact we will establish two local weighted Moser inequalities, one will be used near the boundary (Theorem 2.6) and the other one away from the boundary (Theorem 2.13). These three key estimates along with a suitable Moser iteration scheme as in [GSC] or [CS] lead to the small time parabolic, up to the boundary, Harnack inequality, see Theorem 2.10. In order for the Moser iteration to work, a crucial role is played by the density Theorem 2.11.

Then, with arguments quite similar to the ones used in [GSC] in the complete Riemannian setting, we deduce from the parabolic Harnack inequality, sharp two-sided heat kernel estimates for small time, see Theorem 2.14. To this end a sharp volume estimate is also needed (see Lemma 2.2).

In the sequel we will use the following local representation of the boundary of Ω. There exist a finite number m of coordinate systems (y_i', y_{iN}), $y_i' = (y_{i1}, \cdots, y_{iN-1})$ and the same number of functions $a_i = a_i(y_i')$ defined on the closures of the $(N-1)$ dimensional cubes $\Delta_i := \{y_i' : |y_{ij}| \leq \beta \text{ for } j = 1, \cdots, N-1\}$, $i \in \{1, \cdots, m\}$ so that for each point $x \in \partial\Omega$ there is at least one i such that $x = (x_i', a_i(x_i'))$. The functions a_i satisfy the Lipschitz condition on $\overline{\Delta}_i$ with a constant $A > 0$ that is

$$|a_i(y_i') - a_i(z_i')| \leq A|y_i' - z_i'|$$

for $y_i', z_i' \in \overline{\Delta}_i$; moreover there exists a positive number $\beta < 1$ such that the set B_i defined for any $i \in \{1, \cdots, m\}$ by the relation $B_i = \{(y_i', y_{iN}) : y_i' \in \Delta_i, a_i(y_i') - \beta < y_{iN} < a_i(y_i') + \beta\}$ satisfy $U_i = B_i \cap \Omega = \{(y_i', y_{iN}) : y_i' \in \Delta_i, a_i(y_i') - \beta < y_{iN} < a_i(y_i')\}$ and $\Gamma_i = B_i \cap \partial\Omega = \{(y_i', y_{iN}) : y_i' \in \Delta_i, y_{iN} = a_i(y_i')\}$.

Finally let us observe that for any $y \in U_i$ one can make use of the following estimate on the distance function $(1+A)^{-1}(a_i(y_i') - y_{iN}) \le d(y) \le (a_i(y_i') - y_{iN})$ (see Corollary 4.8 in [K] for details)

Let us fix from now on a constant $\gamma \in (1,2)$ and let us define the "balls" we will use in Moser iteration technique. Roughly speaking they will be Euclidean balls if they stay away from the boundary and they will be N dimensional "deformed cubes", following the geometry of the boundary, if they are close enough to the boundary or even if they intersect it. More precisely we have

Definition 2.1 *(i) For any $x \in \Omega$ and for any $0 < r < \beta$ we define the "ball" centered at x and having radius r as follows. $\mathcal{B}(x,r) = B(x,r)$ the Euclidean ball centered at x and having radius r if $d(x) \ge \gamma r$, while $\mathcal{B}(x,r) = \{(y_i', y_{iN}) : |y_i' - x_i'| \le r,\ a_i(y_i') - r - d(x) < y_{iN} < a_i(y_i') + r - d(x)\}$ if $d(x) < \gamma r$, where $i \in \{1, \cdots, m\}$ is uniquely defined by the point $\bar{x} \in \partial\Omega$ such that $|\bar{x} - x| = d(x)$, that is by the projection of the center x onto $\partial\Omega$. (ii) We also define by $V(x,r) := \int_{\mathcal{B}(x,r)\cap\Omega} |y|^\lambda d^\alpha(y) dy$ the volume of the "ball" centered at x and having radius r.*

We first derive a sharp volume estimate.

Lemma 2.2 *Let $\alpha > 0$, $N \ge 2$, $\lambda \in (-N, 0]$ and $\Omega \subset \mathbf{R}^N$ be a smooth bounded domain containing the origin. Then there exist positive constants c_1, c_2 and β such that for any $x \in \Omega$ and $0 < r < \beta$, we have*

$$c_1 \max\{d^\alpha(x)(|x|+r)^\lambda, r^\alpha\} r^N \le V(x,r) \le c_2 \max\{d^\alpha(x)(|x|+r)^\lambda, r^\alpha\} r^N \ .$$

To this end we make use of the following Lemma which can be proved as in [MT2].

Lemma 2.3 *Let $N \ge 2$, $\lambda \in (-N, 0]$ and $\Omega \subset \mathbf{R}^N$ be a smooth bounded domain containing the origin. Then there exist two positive constants d_1, d_2 such that for any $x \in \Omega$, we have*

$$d_1\ r^N(|x|+r)^\lambda \le \int_{B(x,r)} |y|^\lambda dy \le d_2\ r^N(|x|+r)^\lambda \ . \tag{2.2}$$

Let us accept (2.2) at the moment and let us prove the sharp volume estimate.

Proof of Lemma 2.2: Let us first consider the case where $d(x) \ge \gamma r$. Then $\mathcal{B}(x,r) = B(x,r) \subset \Omega$. Due to the fact that any $y \in B(x,r)$ satisfies

$$\left(\frac{\gamma-1}{\gamma}\right) d(x) \le d(x) - r \le d(y) \le d(x) + r \le \left(\frac{\gamma+1}{\gamma}\right) d(x) \tag{2.3}$$

the claim easily follows making use of Lemma 2.3 with $c_2 \ge d_2 \left(\frac{\gamma+1}{\gamma}\right)^\alpha$ and $c_1 \le d_1 \left(\frac{\gamma-1}{\gamma}\right)^\alpha$.

Let us now consider the case where $d(x) < \gamma r$ and let us denote by L_1, L_2 two positive constants such that $B(0,L_1) \subset \Omega \subset B(0,L_2)$ (note that they exist by assumption on Ω). Then any $y \in \mathcal{B}(x,r) \cap \Omega$ satisfies the following estimate $L_1 - (\gamma+1)\beta \le |y| \le L_2$. Indeed, if on the contrary $|y| < L_1 - (\gamma+1)\beta$, then by definition of L_1 we would have $d(y) > (\gamma+1)\beta$, and this contradicts our assumption. In fact one obviously has $d(y) \le d(x) + r < (\gamma+1)r < (\gamma+1)\beta$ and it is not restrictive to suppose from the beginning that the parameter β in the local representation of the boundary of Ω satisfies $\beta < L_1(\gamma+1)^{-1}$. As a consequence we have:

$$\forall\ y \in \mathcal{B}(x,r) \cap \Omega \quad d_3 \le |y|^\lambda \le d_4 \ . \tag{2.4}$$

here $d_3 := L_2^\lambda$ and $d_4 := (L_1 - (\gamma+1)\beta)^\lambda$.

Then for some $i \in \{1, \cdots, m\}$, we have

$$V(x,r) = \int_{\mathcal{B}(x,r)\cap\Omega} |y|^\lambda d^\alpha(y) dy \sim \int_{|y_i'-x_i'| \le r} \int_{a_i(y_i')-r-d(x)}^{\min\{a_i(y_i'), a_i(y_i')+r-d(x)\}} |y|^\lambda (a_i(y_i') - y_{iN})^\alpha dy_{iN} dy_i' \ .$$

From now on we omit the subscript i for convenience. Indeed we have

$$V(x,r) \le \int_{|y'-x'|\le r} \int_{a(y')-r-d(x)}^{\min\{a(y'),a(y')+r-d(x)\}} |y|^\lambda (a(y')-y_N)^\alpha dy_N dy' \le d_4(\gamma+1) r (d(x)+r)^\alpha \int_{|y'-x'|\le r} dy' \le$$

$$\le d_4(\gamma+1)^{\alpha+1} r^{\alpha+1+N-1} \omega_{N-1} = d_4(\gamma+1)^{\alpha+1} r^{\alpha+N} \omega_{N-1} \ .$$

On the other hand

$$V(x,r) \ge (1+A)^{-\alpha} \int_{|y'-x'|\le r} \int_{a(y')-r-d(x)}^{a(y')-r(\gamma-1)} |y|^\lambda (a(y')-y_N)^\alpha dy_N dy' \ge$$

$$\ge d_3(1+A)^{-\alpha} \int_{|y'-x'|\le r} \int_{a(y')-r-d(x)}^{a(y')-r(\gamma-1)} (r(\gamma-1))^\alpha dy_N dy' \ge$$

$$\ge d_3(1+A)^{-\alpha} r^{\alpha+1} (2-\gamma)(\gamma-1)^\alpha \int_{|y'-x'|\le r} dy' = d_3(1+A)^{-\alpha} r^{\alpha+1+N-1} (2-\gamma)(\gamma-1)^\alpha \omega_{N-1} =$$

$$= d_3(1+A)^{-\alpha} r^{\alpha+N} (2-\gamma)(\gamma-1)^\alpha \omega_{N-1} \ .$$

Here ω_N denotes the standard volume of the Euclidean unit ball in $\mathbf{R}^N$. Thus the result follows with $c_1 := \min\left\{ d_1 \left(\frac{\gamma-1}{\gamma}\right)^\alpha, d_3(1+A)^{-\alpha}(2-\gamma)(\gamma-1)^\alpha \omega_{N-1} \right\}$ and $c_2 := \max\left\{ d_2 \left(\frac{\gamma+1}{\gamma}\right)^\alpha, d_4(\gamma+1)^{\alpha+1}\omega_{N-1} \right\}$.

//

Let us now prove estimate (2.2), which is taken from [MT2], we give here the details for the convenience of the reader

Proof of Lemma 2.3: (i) Observe that

$$\int_{B(x,r)} |y|^\lambda dy = r^{\lambda+N} \int_{B(0,1)} |w+z|^\lambda dz \ ,$$

where $w := \frac{x}{r}$. Then (2.2) reads

$$d_1(|w|+1)^\lambda \le \int_{B(0,1)} |w+z|^\lambda dz \le d_2(|w|+1)^\lambda \ . \tag{2.5}$$

Since $|z| \le 1$, there holds

$$|w|-1 \le |w|-|z| \le |w+z| \le |w|+|z| \le |w|+1 \ ,$$

whence

$$\omega_N \Big(|w|+1\Big)^\lambda \le \int_{B(0,1)} |w+z|^\lambda dz \le \omega_N \Big(|w|-1\Big)^\lambda \ ; \tag{2.6}$$

Comparing (2.5) and (2.6), we immediately obtain the lower bound in (2.2) with $d_1 := \omega_N$. (ii) To prove the upper bound in (2.2), observe that three cases are possible: (a) $r \le \frac{|x|}{2}$, (b) $\frac{|x|}{2} < r < 3|x|$ and (c) $r \ge 3|x|$. In case (a) the claim follows from the right-hand inequality in (2.6), if we exhibit $d_2 > 0$ such that

$$\omega_N(|w|-1)^\lambda \le d_2(|w|+1)^\lambda$$

for any $|w| \ge 2$. It is easily seen that the function $F(t) := \left(\frac{t+1}{t-1}\right)^{|\lambda|}$ $(t > 1)$ is decreasing, thus $F(t) \le F(2) = \left(\frac{1}{3}\right)^\lambda$ for any $t \ge 2$. This proves the claim in this case for any $d_2 \ge \omega_N \frac{1}{3^\lambda}$.

To deal with cases $(b)-(c)$, observe first that

$$\int_{B(0,r)} |y|^\lambda dy = \frac{\omega_N}{\lambda+N} r^{\lambda+N} .$$

In case (b) we have

$$B(x,r) \subseteq B\left(0, 4|x|\right) , \tag{2.7}$$

whence

$$\int_{B(x,r)} |y|^\lambda dy \le \frac{\omega_N}{\lambda+N}(4|x|)^{\lambda+N} \le \frac{\omega_N}{\lambda+N}(8r)^{\lambda+N} . \tag{2.8}$$

In case (c) there holds

$$B(x,r) \subseteq B\left(0, \frac{4r}{3}\right) , \tag{2.9}$$

thus

$$\int_{B(x,r)} |y|^\lambda dy \le \frac{\omega_N}{\lambda+N}\left(\frac{4r}{3}\right)^{\lambda+N} . \tag{2.10}$$

Since

$$r^{\lambda+N} \le \frac{r^N(|x|+r)^\lambda}{3^\lambda} \qquad \text{for } |x| < 2r$$

we have

$$\int_{B(x,r)} |y|^\lambda dy \le \frac{\omega_N}{\lambda+N}\frac{8^{\lambda+N}}{3^\lambda} r^N (|x|+r)^\lambda$$

in cases (b)-(c). Hence the conclusion follows with $d_2 := \frac{\omega_N}{\lambda+N}\frac{8^{\lambda+N}}{3^\lambda}$.

//

From Lemma 2.2 one can easily deduce the doubling property which reads as follows:

Corollary 2.4 *Let $\alpha > 0$, $N \ge 2$, $\lambda \in (-N, 0]$ and $\Omega \subset \mathbf{R}^N$ be a smooth bounded domain containing the origin. Then there exist positive constants C_D and β such that for any $x \in \Omega$ and $0 < r < \beta$, we have*

$$V(x, 2r) \le C_D V(x,r).$$

Let us state now the local Poincaré inequality.

Theorem 2.5 (Local weighted Poincaré inequality) *Let $\alpha > 0$, $N \ge 2$, $\lambda \in (-N, 0]$ and $\Omega \subset \mathbf{R}^N$ be a smooth bounded domain containing the origin. Then there exist positive constants C_P and β such that for any $x \in \Omega$ and $0 < r < \beta$, we have*

$$\inf_{\xi \in \mathbf{R}} \int_{\mathcal{B}(x,r)\cap\Omega} |y|^\lambda d^\alpha(y)|f(y)-\xi|^2 dy \le C_P\, r^2 \int_{\mathcal{B}(x,r)\cap\Omega} |y|^\lambda d^\alpha(y)|\nabla f|^2 dy , \quad \forall\, f \in C^1(\overline{\mathcal{B}(x,r)\cap\Omega}) . \tag{2.11}$$

Proof: Let us first consider the case where $d(x) \ge \gamma r$. Then $\mathcal{B}(x,r) = B(x,r) \subset \Omega$. Due to (2.3) the claim corresponds to Theorem 3.1 in [MT2]. We give here the details for the convenience of the reader.

(i) As a consequence of the compact embedding of the space $H^1(B(0,1), |y|^\lambda dy)$ into $L^2(B(0,1), |y|^\lambda dy)$ (e.g. see [KO]) we have that

$$\int_{B(0,1)} |f-\hat{f}|^2 |y|^\lambda dy \le C \int_{B(0,1)} |\nabla f|^2 |y|^\lambda dy , \quad \forall\, f \in C^1(\overline{B(0,1)}) ;$$

here $\hat{f} := \left(\int_{B(0,1)} f(y)|y|^\lambda dy\right)\left(\int_{B(0,1)} |y|^\lambda dy\right)^{-1}$. Then by scaling (2.11) follows when $x = 0$.

(ii) Let us now consider the case $|x| \geq 2r$ and let us define $\bar{f} := \omega_N^{-1} \int_{B(0,1)} f(x+rz)dz$. Then as in the proof of Lemma 2.3 we have $(|x|+r)^\lambda \leq |x+rz|^\lambda \leq \frac{1}{3^\lambda}(|x|+r)^\lambda$. Hence

$$\int_{B(x,r)} |f-\bar{f}|^2|y|^\lambda dy = r^N \int_{B(0,1)} |f(x+rz)-\bar{f}|^2|x+rz|^\lambda dz \leq \frac{r^N}{3^\lambda}(|x|+r)^\lambda \int_{B(0,1)} |f(x+rz)-\bar{f}|^2 dz \leq$$

$$\leq Cr^2 \frac{r^N}{3^\lambda}(|x|+r)^\lambda \int_{B(0,1)} |\nabla f|^2(x+rz)dz \leq Cr^2 \frac{r^N}{3^\lambda} \int_{B(0,1)} |\nabla f|^2|x+rz|^\lambda dz = Cr^2 \frac{1}{3^\lambda} \int_{B(x,r)} |\nabla f|^2|y|^\lambda dy \,.$$

Then (2.11) follows when $|x| \geq 2r$.

(iii) For a general $x \in \Omega$ two cases are possible (a) $0 \leq |x| < \frac{r}{4}$; (b) $|x| \geq \frac{r}{4}$. In case (a) there holds

$$B\left(x, \frac{r}{8}\right) \subseteq B\left(0, \frac{r}{2}\right) \subseteq B(x,r) \,,$$

thus from (i) we have

$$\inf_{\xi \in \mathbf{R}} \int_{B\left(x,\frac{r}{8}\right)} |f(y)-\xi|^2|y|^\lambda dy \leq \inf_{\xi \in \mathbf{R}} \int_{B\left(0,\frac{r}{2}\right)} |f(y)-\xi|^2|y|^\lambda dy \leq \frac{C}{4} r^2 \int_{B\left(0,\frac{r}{2}\right)} |\nabla f|^2|y|^\lambda dy \leq C_P r^2 \int_{B(x,r)} |\nabla f|^2|y|^\lambda dy \,,$$

This proves (2.11) in case (a) since using a Whitney type covering and arguing as in [SC2] the integration set of the left hand side which from above is $B\left(x, \frac{r}{8}\right)$ can be increased as to cover all $B(x,r)$.

In case (b) there holds $|x| \geq 2\left(\frac{r}{8}\right)$; hence from (ii)

$$\inf_{\xi \in \mathbf{R}} \int_{B\left(x,\frac{r}{8}\right)} |f(y)-\xi|^2|y|^\lambda dy \leq \frac{C}{64} r^2 \int_{B\left(x,\frac{r}{8}\right)} |\nabla f|^2|y|^\lambda dy \leq C_P r^2 \int_{B(x,r)} |\nabla f|^2|y|^\lambda dy \,.$$

This completes the proof in the case $d(x) \geq \gamma r$.

Let us now consider the case where $d(x) < \gamma r$. Then for some $i \in \{1, \cdots, m\}$ we have

$$\int_{\mathcal{B}(x,r)\cap\Omega} d^\alpha(y)|f(y)-\xi|^2 dy \leq \int_{|y_i'-x_i'|\leq r} \int_{a_i(y_i')-r-d(x)}^{\min\{a_i(y_i'),a_i(y_i')+r-d(x)\}} |f(y_i', y_{iN})-\xi|^2|y|^\lambda (a_i(y_i')-y_{iN})^\alpha dy_{iN} dy_i'$$

From now on we omit the subscript i for convenience. Let us perform then the following change of variables $(y', y_N) \to (y', z_N := a(y')-y_N)$ and make use of (2.4); thus the above integral is less or equal then:

$$d_4 \int_{|y'-x'|\leq r} \int_{\max\{0,d(x)-r\}}^{r+d(x)} |f(y', a(y')-z_N)-\xi|^2 z_N^\alpha \; dz_N dy' =$$

$$= d_4 \int_{\max\{0,d(x)-r\}}^{r+d(x)} z_N^\alpha \left(\int_{|y'-x'|\leq r} |f(y', a(y')-z_N)-\xi|^2 dy' \right) dz_N \leq$$

$$\leq C d_4 r^2 \int_{\max\{0,d(x)-r\}}^{r+d(x)} z_N^\alpha \left(\int_{|y'-x'|\leq r} \left| \frac{\partial f}{\partial y'} + \frac{\partial f}{\partial y_n} \frac{a(y')}{\partial y'} \right|^2 dy' \right) dz_N \leq$$

$$\leq C d_4 r^2 \int_{\max\{0,d(x)-r\}}^{r+d(x)} \int_{|y'-x'|\leq r} |\nabla f|^2(y', a(y')-z_N) z_N^\alpha \; dy' dz_N \leq C \frac{d_4}{d_3}(1+A)^\alpha r^2 \int_{\mathcal{B}(x,r)\cap\Omega} |y|^\lambda d^\alpha(y)|\nabla f|^2 dy \,.$$

In the above argument we made the following choices

$$\xi = \xi(z_N) := \left(\int_{|y'-x'|\leq r} f(y', a(y')-z_N) dy' \right) r^{-N+1} \omega_{N-1}^{-1} = \omega_{N-1}^{-1} \int_{|z'-x'|\leq 1} f(rz', a(rz')-z_N) dz' \,,$$

and C being the Euclidean Poincaré constant on the $N-1$ dimensional Euclidean ball of radius one.

Since for any $\bar{\xi} \in \mathbf{R}$, $|f-\bar{\xi}|^2 \le 2|f-\xi(z_N)|^2 + 2|\xi(z_N)-\bar{\xi}|^2$ in order to prove (2.11) in this case, it only remains to estimate the following term

$$\int_{|y'-x'|\le r}\int_{\max\{0,d(x)-r\}}^{r+d(x)} |\xi(z_N)-\bar{\xi}|^2 z_N^\alpha \; dz_N dy' =$$

$$= \left(\int_{|y'-x'|\le r} dy'\right)\left[|\xi(z_N)-\bar{\xi}|^2 \frac{z_N^{\alpha+1}-\max\{0,d(x)-r\}^{\alpha+1}}{\alpha+1}\Big|_{\max\{0,d(x)-r\}}^{r+d(x)} - \right.$$

$$\left. -\frac{2}{\alpha+1}\int_{\max\{0,d(x)-r\}}^{r+d(x)} (\xi(z_N)-\bar{\xi})\frac{\partial \xi(z_N)}{\partial z_N}\left(z_N^{\alpha+1}-\max\{0,d(x)-r\}^{\alpha+1}\right) dz_N\right] .$$

Thus, choosing $\bar{\xi} := \xi(r+d(x))$ above, we obtain by Hölder inequality

$$\int_{|y'-x'|\le r}\int_{\max\{0,d(x)-r\}}^{r+d(x)} |\xi(z_N)-\bar{\xi}|^2 z_N^\alpha \; dz_N dy' \le$$

$$\le \frac{2}{\alpha+1}\left(\int_{|y'-x'|\le r}\int_{\max\{0,d(x)-r\}}^{r+d(x)} |\xi(z_N)-\bar{\xi}|^2 z_N^\alpha \; dz_N dy'\right)^{\frac{1}{2}} \times$$

$$\times \left(\int_{|y'-x'|\le r}\int_{\max\{0,d(x)-r\}}^{r+d(x)} \left|\frac{\partial \xi(z_N)}{\partial z_N}\right|^2 z_N^\alpha \left(\frac{z_N^{\alpha+1}-\max\{0,d(x)-r\}^{\alpha+1}}{z_N^\alpha}\right)^2 dz_N dy'\right)^{\frac{1}{2}} .$$

Since

$$\left|\frac{\partial \xi(z_N)}{\partial z_N}\right| \le \omega_{N-1}^{-1}\int_{|z'-x'|\le 1}\left|\frac{\partial f}{\partial y_N}\right|(rz',a(rz')-z_N)dz' = \omega_{N-1}^{-1} r^{-N+1}\int_{|y'-x'|\le r}\left|\frac{\partial f}{\partial y_N}\right|(y',a(y')-z_N)dy' ,$$

hence

$$\left|\frac{\partial \xi(z_N)}{\partial z_N}\right|^2 \le \omega_{N-1}^{-1} r^{1-N}\int_{|y'-x'|\le r}\left|\frac{\partial f}{\partial y_N}\right|^2(y',a(y')-z_N)dy' .$$

Thus, since $d(x) < \gamma r$, we obtain (2.11) with constant $C_P := 2\frac{d_4}{d_3}(1+A)^\alpha\left(C + \frac{4(\gamma+1)^2}{(\alpha+1)^2}\right)$.

//

We next prove the following local weighted Moser inequality:

Theorem 2.6 (Local weighted Moser inequality) *Let $\alpha > 0$, $N \ge 2$ and $\Omega \subset \mathbf{R}^N$ be a smooth bounded domain. Then there exist positive constants C_M and $R = R(\alpha,\Omega)$ such that for any $\nu \ge N+\alpha$, $x \in \Omega$, $0 < r < R$ and $f \in C_0^\infty(\mathcal{B}(x,r))$ we have*

$$\int_{\mathcal{B}(x,r)\cap\Omega} d^\alpha(y)|f(y)|^{2\left(1+\frac{2}{\nu}\right)}dy \le C_M r^2 V(x,r)^{-\frac{2}{\nu}}\left(\int_{\mathcal{B}(x,r)\cap\Omega} d^\alpha(y)|\nabla f|^2 dy\right)\left(\int_{\mathcal{B}(x,r)\cap\Omega} d^\alpha(y)|f|^2 dy\right)^{\frac{2}{\nu}} .$$

Proof: Let us first consider the case where $d(x) \ge \gamma r$. By the standard Moser inequality, there exists a positive constant C such that for any $x \in \Omega$ and any $\nu \ge N$ if $N \ge 3$ or any $\nu > 2$ if $N = 2$, the following holds true

$$\int_{B(x,r)} |f(y)|^{2\left(1+\frac{2}{\nu}\right)}dy \le C r^2 r^{-\frac{2N}{\nu}}\left(\int_{B(x,r)} |\nabla f|^2 dy\right)\left(\int_{B(x,r)} |f|^2 dy\right)^{\frac{2}{\nu}} \quad , \quad \forall\ f \in C_0^\infty(B(x,r))$$

(see for example Section 2.1.3 in [SC2]). Thus we have

$$\int_{B(x,r)} d^\alpha(y)|f(y)|^{2\left(1+\frac{2}{\nu}\right)}dy \leq (d(x)+r)^\alpha C r^2 r^{-\frac{2N}{\nu}} \left(\int_{B(x,r)} |\nabla f|^2 dy\right)\left(\int_{B(x,r)} |f|^2 dy\right)^{\frac{2}{\nu}} \leq$$

$$\leq C r^2 \left(\frac{d(x)+r}{d(x)-r}\right)^\alpha (r^N(d(x)-r)^\alpha)^{-\frac{2}{\nu}} \left(\int_{B(x,r)} d^\alpha(y)|\nabla f|^2 dy\right)\left(\int_{B(x,r)} d^\alpha(y)|f|^2 dy\right)^{\frac{2}{\nu}} \leq$$

$$\leq C_M r^2 V(x,r)^{-\frac{2}{\nu}} \left(\int_{B(x,r)} d^\alpha(y)|\nabla f|^2 dy\right)\left(\int_{B(x,r)} d^\alpha(y)|f|^2 dy\right)^{\frac{2}{\nu}} ,$$

where $C_M := C\left(1+\frac{2}{\gamma-1}\right)^\alpha \left(\frac{\gamma}{\gamma-1}\right)^{\frac{2\alpha}{\nu}} c_2^{\frac{2}{\nu}}$ and c_2 is the constant appearing in the volume estimate in Lemma 2.2 when $\lambda = 0$.

Let us now consider the case where $d(x) < \gamma r$. Then we claim the following local weighted Sobolev inequality: there exist positive constants C_S and $R = R(\alpha, \Omega)$ such that for any $x \in \Omega$, $0 < r < R$, satisfying $d(x) < \gamma r$, and any $f \in C_0^\infty(\mathcal{B}(x,r))$, we have

$$\left(\int_{\mathcal{B}(x,r)\cap\Omega} d^\alpha(y)|f(y)|^{\frac{2(N+\alpha)}{N+\alpha-2}} dy\right)^{\frac{N+\alpha-2}{N+\alpha}} \leq C_S \int_{\mathcal{B}(x,r)\cap\Omega} d^\alpha(y)|\nabla f|^2 dy . \tag{2.12}$$

If we accept (2.12), then the result follows, with $C_M = C_S$ by means of Hölder inequality, in fact we have

$$\int_{\mathcal{B}(x,r)\cap\Omega} d^\alpha(y) f^{2\left(1+\frac{2}{N+\alpha}\right)} dy \leq \left(\int_{\mathcal{B}(x,r)\cap\Omega} d^\alpha(y) f^{\frac{2(N+\alpha)}{N+\alpha-2}} dy\right)^{\frac{N+\alpha-2}{N+\alpha}} \left(\int_{\mathcal{B}(x,r)\cap\Omega} d^\alpha(y) f^2 dy\right)^{\frac{2}{N+\alpha}}$$

as well as for any $\nu > N+\alpha$

$$\left(\int_{\mathcal{B}(x,r)\cap\Omega} d^\alpha(y) f^{2\left(1+\frac{2}{\nu}\right)} dy\right)\left(\int_{\mathcal{B}(x,r)\cap\Omega} d^\alpha(y) f^2 dy\right)^{\frac{2(\nu-N-\alpha)}{\nu(N+\alpha)}} \leq \left(\int_{\mathcal{B}(x,r)\cap\Omega} d^\alpha(y) f^{2\left(1+\frac{2}{N+\alpha}\right)} dy\right) V(x,r)^{\frac{2(\nu-N-\alpha)}{\nu(N+\alpha)}} .$$

In the sequel we will give the proof of (2.12). We will follow closely the argument of [FMT2]. If $V \subset \mathbf{R}^N$ is any bounded domain and $u \in C^\infty(\overline{V})$ then it is well known that

$$S_N ||u||_{L^{\frac{N}{N-1}}(V)} \leq ||\nabla u||_{L^1(V)} + ||u||_{L^1(\partial V)} ,$$

where $S_N := N\pi^{\frac{1}{2}}\left[\Gamma(1+\frac{N}{2})\right]^{-\frac{1}{2}}$ (see p. 189 in [M]). Let us fix from now on that $V := \mathcal{B}(x,r)\cap\Omega$, and let us apply the above inequality to $u := d^a f$, for any $f \in C_0^\infty(\mathcal{B}(x,r))$ and any $a > 0$. Thus we get

$$S_N ||d^a f||_{L^{\frac{N}{N-1}}(V)} \leq \int_V \left(|\nabla f| d^a + a d^{a-1}|\nabla d||f|\right) dy .$$

Let us remark at this point that boundary terms on $\partial\Omega$ are zero due to the presence of the weight d^a, $a > 0$. To estimate the last term of the right hand side, we will make use of an integration by parts, noting that $\nabla d \cdot \nabla d = 1$ a.e.; that is we have:

$$a\int_V d^{a-1}|f| dy = a\int_V \nabla d\cdot\nabla d\ d^{a-1}|f| dy = \int_V \nabla d^a \cdot \nabla d |f| dy =$$

$$= -\int_V d^a \Delta d|f|dy - \int_V d^a \nabla d \cdot \nabla |f|dy + \int_{\partial V} d^a \nabla d \cdot \nu \ |f| \ dS.$$

Under our smoothness assumption on Ω we have that $|d\Delta d| \leq c_0 \delta$ in Ω_δ for δ small, say $0 < \delta \leq \delta_0$, and for some positive constant c_0 independent of δ (δ_0, c_0 depending on Ω). Now, if $d(x) + r < \delta$, that is if $r < \frac{\delta}{\gamma+1}$, we have that $V \subset \Omega_\delta$ and it follows that

$$a \int_V d^{a-1}|f|dy \leq c_0 \delta \int_V d^{a-1}|f|dy + \int_V d^a |\nabla f|dy \ ,$$

hence

$$\int_V d^{a-1}|f|dy \leq (a - c_0\delta)^{-1} \int_V d^a |\nabla f|dy \ . \tag{2.13}$$

Consequently for any $r \in (0, R(a,\Omega))$, $R(a,\Omega) := \frac{1}{\gamma+1} \min\{\delta_0, \frac{a}{c_0}\}$ and any $a > 0$ the following inequality is true

$$S_N ||d^a f||_{L^{\frac{N}{N-1}}(V)} \leq \left(\frac{a}{a - c_0 \delta_0} + 1 \right) \int_V d^a |\nabla f|dy \ . \tag{2.14}$$

To proceed we will use the following interpolation inequality (cf. Lemma 4.1 of [FMT2]).

$$||d^b f||_{L^q(V)} \leq \frac{N(q-1)}{q} ||d^a f||_{L^{\frac{N}{N-1}}(V)} + \frac{q - N(q-1)}{q} ||d^{a-1} f||_{L^1(V)} \ ,$$

$$\forall \ 1 < q \leq \frac{N}{N-1} \ , \quad b := a - 1 + \frac{q-1}{q} N \ , \quad a > 0 \ . \tag{2.15}$$

From (2.13) and (2.14), we get for any a, b, q as above the following inequality

$$||d^b f||_{L^q(V)} \leq C_1 ||d^a \nabla f||_{L^1(V)} \ , \tag{2.16}$$

where $C_1 := \frac{N(q-1)}{q} \frac{1}{S_N} \left(\frac{a}{a - c_0\delta_0} + 1 \right) + \frac{q - N(q-1)}{q} \left(\frac{1}{a - c_0 \delta_0} \right)$.

Let us now apply inequality (2.16) to $|f|^s$ instead of f, for $s := \frac{Q}{2} + 1$, $q := \frac{Q}{s}$, $b := Bs$. Due to (2.15) we have $a = b + 1 - \frac{q-1}{q} N = \frac{BQ}{2} + A$, where $A := B + 1 - \frac{Q-2}{2Q} N$. In this way we obtain

$$\left(\int_V d^{BQ} |f|^Q dy \right)^{\left(\frac{Q}{2}+1\right)\frac{1}{Q}} \leq C_1 \left(\frac{Q}{2} + 1 \right) \left(\int_V d^{\frac{BQ}{2}+A} |f|^{\frac{Q}{2}} |\nabla f| dy \right) \leq$$

$$\leq C_2 \left(\int_V d^{BQ} |f|^Q dy \right)^{\frac{1}{2}} \left(\int_V d^{2A} |\nabla f|^2 dy \right)^{\frac{1}{2}} \ ;$$

where $C_2 := C_1 \left(\frac{Q}{2} + 1 \right)$.

After simplifying we see that we have proved the following: there exists $R = R(\frac{BQ}{2} + A, \Omega)$ such that for all $0 < r < R$ and all $x \in \Omega$ with $d(x) < \gamma r$, there holds

$$\left(\int_{\mathcal{B}(x,r)\cap\Omega} d^{BQ} |f|^Q dy \right)^{\frac{2}{Q}} \leq C \int_{\mathcal{B}(x,r)\cap\Omega} d^{2A}(y) |\nabla f|^2 dy \ ,$$

for any $N \geq 2$ and any $f \in C_0^\infty(\mathcal{B}(x,r))$ under the following conditions $A := B + 1 - \frac{Q-2}{2Q} N$, $\frac{BQ}{2} + A > 0$, $2 < Q < \infty$ if $N = 2$, $2 < Q \leq \frac{2N}{N-2}$ if $N \geq 3$; here $C_3 = C_2^2 = C_3(N, Q, B, c_0, \delta_0)$.

Taking $A = \frac{\alpha}{2}$, $Q := \frac{2(N+\alpha)}{N+\alpha-2}$ and $B := \frac{\alpha}{Q}$ we deduce the local weighted Sobolev inequality (2.12) with $C_S = C_S(N, \alpha, c_0, \delta_0)$ and this completes the proof of Theorem 2.6.

//

Remark 2.7 *Note that the upper bound for the length of the "balls" in the local weighted Moser inequality, denoted by $R(\alpha,\Omega)$, goes to zero as α tends to zero.*

Remark 2.8 *Let us note that when $N=1$, the corresponding analogue of the local weighted Sobolev inequality (2.12) when $\Omega=(-1,1)$ is the following one*

$$\left(\int_{\max\{-1,x-r\}}^{\min\{1,x+r\}}(1-|y|)^{\alpha}|f|^{q}(y)dy\right)^{\frac{1}{q}} \le C_S\ r^{\frac{\alpha+1}{q}+\frac{1-\alpha}{2}}\left(\int_{\max\{-1,x-r\}}^{\min\{1,x+r\}}(1-|y|)^{\alpha}|f'|^{2}(y)dy\right)^{\frac{1}{2}} ,$$

for any $f\in C_0^{\infty}(x-r,x+r)$, and any $q>2$ if $0<\alpha\le 1$ and $2<q\le\frac{2(\alpha+1)}{\alpha-1}$ if $\alpha>1$. Consequently Theorem 1.5 as well as its consequences can be also stated for $N=1$; see [KO]

From the results within this subsection, we will now deduce a new parabolic Harnack inequality up to the boundary for the doubly degenerate elliptic operator L_α^λ defined in (2.1). To this end let us first make precise the notion of a weak solution

Definition 2.9 *By a solution $v(y,t)$ to $v_t=-L_\alpha^\lambda v$ in $Q:=\{\mathcal{B}(x,r)\cap\Omega\}\times(0,r^2)$, we mean a function $v\in C^1((0,r^2);L^2(\mathcal{B}(x,r)\cap\Omega,|y|^\lambda\ d^\alpha(y)dy))\cap C^0((0,r^2);H^1(\mathcal{B}(x,r)\cap\Omega,|y|^\lambda\ d^\alpha(y)dy))$ such that for any $\Phi\in C^0((0,r^2);C_0^\infty(\mathcal{B}(x,r)\cap\Omega))$ and any $0<t_1<t_2<r^2$ we have*

$$\int_{t_1}^{t_2}\int_{\mathcal{B}(x,r)\cap\Omega}\{|y|^\lambda d^\alpha(y)v_t\Phi+|y|^\lambda d^\alpha(y)\nabla v\nabla\Phi\}dydt=0\ . \tag{2.17}$$

Then we have

Theorem 2.10 *Let $\alpha\ge 1$, $N\ge 2$, $\lambda\in[2-N,0]$ and $\Omega\subset\mathbf{R}^N$ be a smooth bounded domain containing the origin. Then there exist positive constants C_H and $R=R(\Omega)$ such that for $x\in\Omega$, $0<r<R$ and for any positive solution $v(y,t)$ of $\frac{\partial v}{\partial t}=\frac{1}{|y|^\lambda d^\alpha(y)}div(|y|^\lambda d^\alpha(y)\nabla v)$ in $\{\mathcal{B}(x,r)\cap\Omega\}\times(0,r^2)$, the following estimate holds true*

$$ess\ sup_{(y,t)\in\left\{\mathcal{B}(x,\frac{r}{2})\cap\Omega\right\}\times(\frac{r^2}{4},\frac{r^2}{2})}v(y,t)\le C_H\ ess\ inf_{(y,t)\in\left\{\mathcal{B}(x,\frac{r}{2})\cap\Omega\right\}\times(\frac{3}{4}r^2,r^2)}v(y,t)\ .$$

In order to prove the parabolic Harnack inequality in Theorem 2.10 we use the Moser iteration technique as adapted to degenerate elliptic operators in [FKS], [CS] as well as [GSC]. In this approach one inserts in the weak form of the equation $v_t=-L_\alpha^\lambda v$ suitable test functions Φ. One of the key ideas is to use test functions Φ of the form η^2v^q, where v is the weak solution of the equation, η is a cut off function and $q\in\mathbf{R}$. To this end one has to check that η^2v^q is in the right space of test function. In this direction the following density theorem is crucial.

Theorem 2.11 *Let $N\ge 2$ and $\Omega\subset\mathbf{R}^N$ be a smooth bounded domain. Then for any $\alpha\ge 1$*

$$H^1(\Omega,\ d^\alpha(y)\ dy)=H_0^1(\Omega,d^\alpha(y)\ dy)\ .$$

In particular for any $\alpha\ge 1$, the set $C_0^\infty(\Omega)$ is dense in $H^1(\Omega,d^\alpha(y)dy)$.

Here $H^1(\Omega,d^\alpha(y)dy)$ denotes the set $\{v=v(y):\int_\Omega d^\alpha(y)(v^2+|\nabla v|^2)dy<\infty\}$, the corresponding norm being defined in (1.12).

We are now ready to prove the density theorem.

Proof Let us prove here the result when $\alpha=1$. We refer to Proposition 9.10 in [K] for the case $\alpha>1$, even though our proof with some minor changes, can also cover this range.

First of all from Theorem 7.2 in [K] it is known that the set $C^\infty(\overline{\Omega})$ is dense in $H^1(\Omega, d(y)\ dy)$. Thus for any $v \in H^1(\Omega, d(y)\ dy)$ there exists $v_m \in C^\infty(\overline{\Omega})$ such that for any $\epsilon > 0$ we have $||v - v_m||_{H^1_1} \le \epsilon$ if $m \ge m(\epsilon)$. Let us choose $w := v_{m(\epsilon)}$ and let us define, for $k \ge 1$, the following function

$$\varphi_k(x) = \begin{cases} 0 & if\ d(x) \le \frac{1}{k^2}\ , \\ 1 + \frac{\ln(kd(x))}{\ln(k)} & if\ \frac{1}{k^2} < d(x) < \frac{1}{k}\ , \\ 1 & if\ d(x) \ge \frac{1}{k}\ . \end{cases}$$

Then $w_k := w\varphi_k \in C_0^{0,1}(\Omega)$, moreover we have

$$||w - w_k||_{H^1_1} = ||w(1-\varphi_k)||_{H^1_1} \le 2\int_\Omega (w^2 + |\nabla w|^2)(1-\varphi_k)^2 d(y)\ dy + 2\int_\Omega w^2 |\nabla \varphi_k|^2 d(y)\ dy \le$$

$$\le 2\int_{d(y)<\frac{1}{k}} (w^2 + |\nabla w|^2) d(y)\ dy + 2\int_{\frac{1}{k^2}<d(y)<\frac{1}{k}} \frac{w^2}{d(y)(\ln(k))^2}\ dy\ .$$

Now as $k \to \infty$ the right hand side goes to zero, this proves the Theorem.

//

The above Theorem allows us to take the cut off function η in $C_0^\infty(\mathcal{B}(x,r))$ instead of taking it as usual in $C_0^\infty(\mathcal{B}(x,r) \cap \Omega)$. Clearly the two function spaces differ only if the “ball” intersects the boundary of Ω. To explain what are the appropriate modifications of the standard iteration argument by Moser, we now present in detail the first step, which is the L^2 mean value inequality for any positive local subsolution of the equation $v_t = -L^\lambda_\alpha v$.

Theorem 2.12 *Let $\alpha \ge 1$, $N \ge 2$, $\lambda \in [2-N, 0]$ and $\Omega \subset \mathbf{R}^N$ be a smooth bounded domain containing the origin. Then there exist positive constants C and $R(\Omega)$ such that for $x \in \Omega$, $0 < r < R(\Omega)$ and for any positive subsolution $v(y,t)$ of $v_t - \frac{1}{|y|^\lambda d^\alpha(y)} div(|y|^\lambda d^\alpha(y)\nabla v) = 0$ in $\{\mathcal{B}(x,r)\cap\Omega\} \times (0, r^2)$ we have the estimate*

$$ess\ sup_{(y,t)\in\{\mathcal{B}(x,\frac{r}{2})\cap\Omega\}\times(\frac{r^2}{2},r^2)}\ \ v^2(y,t) \le \frac{C}{r^2 V(x,r)} \int_{\{\mathcal{B}(x,r)\cap\Omega\}\times(0,r^2)} |y|^\lambda d^\alpha(y)\ v^2(y,t) dydt\ .$$

Proof: We will only prove the result in the non standard case in which the “ball” $\mathcal{B}(x,r)$ intersects the boundary of Ω; we refer to [MT2] as well as to [GSC] for details in the other case. Similarly to Definition 2.9 we define a subsolution $v(y,t)$ to be a function in $C^1((0,r^2); L^2(\mathcal{B}(x,r)\cap\Omega, |y|^\lambda d^\alpha(y)\ dy)) \cap C^0((0,r^2); H^1(\mathcal{B}(x,r)\cap\Omega, |y|^\lambda d^\alpha(y)\ dy))$ such that the following holds true

$$\int_0^{r^2} \int_{\mathcal{B}(x,r)\cap\Omega} \{|y|^\lambda d^\alpha(y) v_t \Phi + |y|^\lambda d^\alpha(y) \nabla v \nabla \Phi\} dydt \le 0, \quad \forall\ \Phi \in C^0((0,r^2); C_0^\infty(\mathcal{B}(x,r)\cap\Omega))\ , \Phi \ge 0.$$

Hence in particular we have also

$$\int_{\mathcal{B}(x,r)\cap\Omega} \{|y|^\lambda d^\alpha(y) v_t \Phi + |y|^\lambda d^\alpha(y) \nabla v \nabla \Phi\} dy \le 0, \quad \forall\ \Phi \in C_0^\infty(\mathcal{B}(x,r)\cap\Omega)\ , \Phi \ge 0.$$

Let us define for any $q, M \ge 1$ the following functions $G(z) = z^q$ if $z \le M$ and $G(z) = M^q + q(z-M)M^{q-1}$ if $z > M$ and $H(z) \ge 0$ by $H'(z) = \sqrt{G'(z)}$, $H(0) = 0$; note that $G(z) \le zG'(z)$ as well as $H(z) \le zH'(z)$. Due to Theorem 2.11 there exists a sequence of functions v_m in $C^\infty(\overline{\mathcal{B}(x,r)\cap\Omega})$ having compact support in Ω such that $v_m \to v$ in $H^1(\mathcal{B}(x,r)\cap\Omega, d^\alpha(y)\ dy)$ as $m \to +\infty$; whence due to (2.4) also in

$H^1(\mathcal{B}(x,r)\cap\Omega, |y|^\lambda d^\alpha(y)\ dy)$. Hence for any $\eta\in C_0^\infty(\mathcal{B}(x,r))$ and $m\geq 1$ the function $\Phi := \eta^2 G(v_m)$ is an admissible test function, that is the following holds true

$$\int_{\mathcal{B}(x,r)\cap\Omega}\{|y|^\lambda d^\alpha(y)\eta^2 G(v_m)v_t + |y|^\lambda d^\alpha(y)\nabla v\nabla(\eta^2 G(v_m))\}dy \leq 0\ .$$

Passing to the limit as $m\to+\infty$ we get

$$\int_{\mathcal{B}(x,r)\cap\Omega}\{|y|^\lambda d^\alpha(y)\eta^2 G(v)v_t + |y|^\lambda d^\alpha(y)\nabla v\nabla(\eta^2 G(v))\}dy \leq 0\ ,\quad \forall\, \eta\in C_0^\infty(\mathcal{B}(x,r))\ .$$

This is the standard starting point in Moser iteration technique apart from the fact that the cut off function η does not be necessarily zero on $\partial\Omega$, this is crucial. Then by Schwarz inequality we get

$$\int_{\mathcal{B}(x,r)\cap\Omega}\{|y|^\lambda d^\alpha(y)\eta^2 G(v)v_t + |y|^\lambda d^\alpha(y)|\nabla v|^2 G'(v)\eta^2\}dy \leq C\int_{\mathcal{B}(x,r)\cap\Omega}|y|^\lambda d^\alpha(y)|\nabla\eta|^2 v^2 G'(v)dy$$

thus also that

$$\int_{\mathcal{B}(x,r)\cap\Omega}\{|y|^\lambda d^\alpha(y)\eta^2 G(v)v_t + |y|^\lambda d^\alpha(y)|\nabla(\eta H(v))|^2\}dy \leq C\int_{\mathcal{B}(x,r)\cap\Omega}|y|^\lambda d^\alpha(y)|\nabla\eta|^2 v^2 G'(v)dy\ .$$

For any smooth function χ of the time variable t, we easily get

$$\frac{d}{dt}\int_{\mathcal{B}(x,r)\cap\Omega}|y|^\lambda d^\alpha(y)(\eta\chi F(v))^2 dy + \chi^2\int_{\mathcal{B}(x,r)\cap\Omega}|y|^\lambda d^\alpha(y)|\nabla(\eta H(v))|^2 dy \leq$$

$$\leq C\chi\left(\chi||\nabla\eta||_{L^\infty(\mathbf{R}^n)} + ||\chi'||_{L^\infty(\mathbf{R})}\right)\int_{supp\eta\ \cap\Omega}|y|^\lambda d^\alpha(y)v^2 G'(v)dy;$$

here $F(z)$ is such that $2F(z)F'(z) = G(z)$. For $\frac{1}{2}\leq s < s' < 1$ we choose as usual χ such that $0\leq\chi\leq 1$, $\chi = 0$ in $(-\infty, r^2(1-s'))$, $\chi = 1$ in $(r^2(1-s),\infty)$, moreover if $\xi\in C_0^\infty(0,1)$ be a nonnegative non increasing function such that $\xi(z) = 1$ if $z\leq s$ and $\xi(z) = 0$ if $z\geq s'$, we define, making use of local coordinates, the following cut off function $\eta(y) := \xi\left(\frac{|y'-x'|}{r}\right)\xi\left(\frac{|a(y')-y_N-d(x)|}{r}\right)$. Then clearly $||\nabla\eta||_{L^\infty(\mathbf{R}^n)}\leq\frac{C}{r(s'-s)}$ and $||\chi'||_{L^\infty(\mathbf{R})}\leq\frac{C}{r^2(s'-s)}$.

Integrating our inequality over $(0,t)$, with $t\in(r^2(1-s), r^2)$ we obtain

$$\sup_{t\in J}\int_{\mathcal{B}(x,r)\cap\Omega}|y|^\lambda d^\alpha(y)(\eta F(v))^2 dy + \int_{\{\mathcal{B}(x,r)\cap\Omega\}\times(r^2(1-s),r^2)}|y|^\lambda d^\alpha(y)|\nabla(\eta H(v))|^2 dydt \leq$$

$$\leq\frac{C}{r^2(s'-s)^2}\int_{\{\mathcal{B}(x,s'r)\cap\Omega\}\times(r^2(1-s'),r^2)}|y|^\lambda d^\alpha(y)v^2 G'(v)dydt\ .$$

Making once again use of Theorem 2.11 we note that we can apply the local weighted Moser inequality in Theorem 2.6 to the function $f := \eta H(v)$ thus obtaining

$$\int_{\{\mathcal{B}(x,r)\cap\Omega\}\times(r^2(1-s),r^2)}|y|^\lambda d^\alpha(y)(\eta H(v))^{2\left(1+\frac{2}{N+\alpha}\right)}dydt \leq$$

$$\leq\frac{C}{r^2(s'-s)^2}\left(\int_{\{\mathcal{B}(x,s'r)\cap\Omega\}\times(r^2(1-s'),r^2)}|y|^\lambda d^\alpha(y)v^2 G'(v)dydt\right)^{1+\frac{2}{N+\alpha}}\ .$$

Let us now denote by $\tilde\gamma := 1+\frac{2}{N+\alpha}$ thus as M tends to infinity we have for $p:=q+1$

$$\int_{\{\mathcal{B}(x,sr)\cap\Omega\}\times(r^2(1-s),r^2)}|y|^\lambda d^\alpha(y)v^{p\tilde\gamma}dydt \le \frac{C}{r^2(s'-s)^2}\left(\int_{\{\mathcal{B}(x,s'r)\cap\Omega\}\times(r^2(1-s'),r^2)}|y|^\lambda d^\alpha(y)v^p dydt\right)^{\tilde\gamma} .$$

Thus due to Lemma 2.2 also that

$$V(x,sr)^{-1}(r^2s)^{-1}\int_{\{\mathcal{B}(x,sr)\cap\Omega\}\times(r^2(1-s),r^2)}|y|^\lambda d^\alpha(y)v^{p\tilde\gamma}dydt \le$$

$$\le C\left(\frac{s}{s'-s}\right)^2\left(V(x,s'r)^{-1}(r^2s')^{-1}\int_{\{\mathcal{B}(x,s'r)\cap\Omega\}\times(r^2(1-s'),r^2)}|y|^\lambda d^\alpha(y)v^p dydt\right)^{\tilde\gamma} .$$

Take now $p=p_i:=2\tilde\gamma^i$, $s=\theta_{i+1}$ and $s'=\theta_i$ where $\theta_i := \frac{i+2}{2(i+1)}$ then if we denote by $I(i) := \Big(V(x,\theta_i r)^{-1}(r^2\theta_i)^{-1}\int_{\{\mathcal{B}(x,\theta_i r)\cap\Omega\}\times(r^2(1-\theta_i),r^2)}|y|^\lambda d^\alpha(y)v^{p_i}dydt\Big)^{\frac{1}{p_i}}$ the above inequality can be restated as follows $I(i+1)\le C(i)I(i)$. Thus since one can show that the product of $C(i)$ for all $i\ge 0$ is finite, we obtain $I(\infty)\le\{\prod_{i=0}^\infty C(i)\}\, I(0)$, this completes the proof of the proposition. To this end the choice $R(\Omega):=\min\{\beta,R(1,\Omega)\}$ can be made, here β and $R(1,\Omega)$ are the constants appearing respectively in the local representation of $\partial\Omega$ and in Theorem 2.6 when $\alpha:=1$.

//

Theorem 2.6 corresponds to the local weighted Moser inequality needed in the proof of the parabolic Harnack inequality up to the boundary stated in Theorem 1.5. The local weighted Moser inequality involved in the proof of Theorem 2.10 differs from Theorem 2.6 only if $d(x)\ge\gamma r$, $N\ge 3$, $\lambda\neq 0$, and in this case it reads as follows

Theorem 2.13 *Let $N\ge 3$, $\lambda\in[2-N,0)$ and $\Omega\subset\mathbf{R}^N$ be a smooth bounded domain containing the origin. Then there exist a positive constant C_M such that for any $\nu\ge N$, $x\in\Omega$, $r>0$ and $f\in C_0^\infty(\mathcal{B}(x,r))$ we have*

$$\int_{B(x,r)}|y|^\lambda|f(y)|^{2\left(1+\frac{2}{\nu}\right)}dy \le C_M r^2(r^N(|x|+r)^\lambda)^{-\frac{2}{\nu}}\left(\int_{B(x,r)}|y|^\lambda|\nabla f|^2dy\right)\left(\int_{B(x,r)}|y|^\lambda|f|^2dy\right)^{\frac{2}{\nu}} .$$

Proof : By Hölder inequality the result easily follows with $C_M:=C_S$ as soon as the following local weighted Sobolev inequality holds true

$$\left(\int_{B(x,r)}|y|^\lambda|f(y)|^{\frac{2N}{N-2}}dy\right)^{\frac{N-2}{N}} \le C_S(|x|+r)^{\frac{2|\lambda|}{N}}\int_{B(x,r)}|y|^\lambda|\nabla f|^2dy \tag{2.18}$$

(we refer to the proof of Theorem 2.6 where a similar argument is used). Let us first prove the above inequality for any $\lambda\in(2-N,0)$. As a consequence of the Caffarelli Kohn Nirenberg inequality (e.g. see Corollary 2 in Section 2.1.6 of [M]), the following holds true

$$\left(\int_{B(x,r)}f^{\frac{2N}{N-2}}|y|^{\frac{N\lambda}{N-2}}dy\right)^{\frac{N-2}{N}} \le C\int_{B(x,r)}|\nabla f|^2|y|^\lambda dy\ ,\quad \forall f\in C_0^\infty(B(x,r))$$

and for some positive constant C independent of x and r. Whence also that

$$\left(\int_{B(x,r)}f^{\frac{2N}{N-2}}|y|^\lambda dy\right)^{\frac{N-2}{N}} \le C\left(\sup_{y\in B(x,r)}|y|\right)^{\frac{2|\lambda|}{N}}\int_{B(x,r)}|\nabla f|^2|y|^\lambda dy \le C(|x|+r)^{\frac{2|\lambda|}{N}}\int_{B(x,r)}|\nabla f|^2|y|^\lambda dy .$$

Let us now prove the result for $\lambda = 2-N$. To this end let us apply Proposition 3.1 to $\Omega = B(0,1)$ with $D = e^{\frac{1}{N-2}}$. Then there exists a positive constant C such that

$$\int_{B(0,1)} |\nabla v|^2 |x|^{2-N} dx \geq C \left(\int_{B(0,1)} v^{\frac{2N}{N-2}} |x|^{-N} X^{\frac{2(N-1)}{N-2}} \left(\frac{|x|}{D} \right) dx \right)^{\frac{N-2}{N}} , \quad \forall\, v \in C_0^\infty(B(0,1)) \; ;$$

here $X(t) = \frac{1}{1-\ln t}$, $t \in (0,1]$. Now let us take $v(x) := f\left(\frac{x}{R}\right)$ for any $f \in C_0^\infty(B(0,R))$ then from above we have

$$\int_{B(0,R)} |\nabla f|^2 |y|^{2-N} dy \geq C \left(\int_{B(0,R)} f^{\frac{2N}{N-2}} |y|^{-N} X^{\frac{2(N-1)}{N-2}} \left(\frac{|y|}{DR} \right) dy \right)^{\frac{N-2}{N}} .$$

Then if $y \in B(x,r)$ clearly $y \in B(0, |x|+r)$, thus if we take $R := |x|+r$ and $f \in C_0^\infty(B(x,r))$ from above we have

$$\int_{B(x,r)} |\nabla f|^2 |y|^{2-N} dy \geq C \left(\int_{B(x,r)} f^{\frac{2N}{N-2}} |y|^{-N} X^{\frac{2(N-1)}{N-2}} \left(\frac{|y|}{DR} \right) dy \right)^{\frac{N-2}{N}} \geq$$

$$\geq \left(\int_{B(x,r)} f^{\frac{2N}{N-2}} |y|^{2-N} dy \right)^{\frac{N-2}{N}} \left(\inf_{y \in B(x,r)} |y|^{-2} X^{\frac{2(N-1)}{N-2}} \left(\frac{|y|}{DR} \right) \right)^{\frac{N-2}{N}} .$$

Whence the claim easily follows as soon as we prove that

$$\left(\sup_{y \in B(x,r)} |y| X^{-\frac{N-1}{N-2}} \left(\frac{|y|}{DR} \right) \right)^{\frac{2(N-2)}{N}} \leq C_S (|x|+r)^{\frac{2(N-2)}{N}} .$$

This is indeed the case in fact we have

$$\sup_{y \in B(x,r)} |y| X^{-\frac{N-1}{N-2}} \left(\frac{|y|}{DR} \right) \leq \sup_{0 \leq |y| \leq |x|+r} |y| \left(1 - \ln \left(\frac{|y|}{DR} \right) \right)^{\frac{N-1}{N-2}} =$$

(thus using the fact that the function $\varphi(t) = t\left(1 - \ln\left(\frac{t}{DR}\right)\right)^{\frac{N-1}{N-2}}$ is an increasing function for $t \in [0,R]$ if D and R are as above)

$$= (|x|+r) \left(1 - \ln \left(\frac{|x|+r}{DR} \right) \right)^{\frac{N-1}{N-2}} = (|x|+r)(1+\ln(D))^{\frac{N-1}{N-2}} = (|x|+r) \left(\frac{N-1}{N-2} \right)^{\frac{N-1}{N-2}} .$$

This completes the proof of Theorem 2.13.

//

To state the heat kernel estimates following from Theorem 2.10 we introduce some notation. The operator L_α^λ is defined for $\alpha \geq 1$ and $\lambda \in [2-N, 0]$ in $L^2(\Omega, |x|^\lambda d^\alpha(x)\, dx)$ as the generator of the symmetric form

$$\mathcal{L}_\alpha^\lambda [v_1, v_2] := \int_\Omega |x|^\lambda d^\alpha(x) \nabla v_1 \nabla v_2 \; dx ,$$

namely

$$D(L_\alpha^\lambda) := \left\{ v \in H_0^1(\Omega, |x|^\lambda d^\alpha(x)\, dx) : -\frac{1}{|x|^\lambda d^\alpha(x)} div(|x|^\lambda d^\alpha(x) \nabla v) \in L^2(\Omega, |x|^\lambda d^\alpha(x)\, dx) \right\},$$

$$L_\alpha^\lambda v := -\frac{1}{|x|^\lambda d^\alpha(x)} div(|x|^\lambda d^\alpha(x) \nabla v) \text{ for any } v \in D(L_\alpha^\lambda) ,$$

where $H_0^1(\Omega, |x|^\lambda d^\alpha(x)\ dx)$ denotes the closure of $C_0^\infty(\Omega)$ in the norm

$$v \to ||v||_{H^1_{\alpha,\lambda}} := \left\{ \int_\Omega |x|^\lambda d^\alpha(x) \left(|\nabla v|^2 + v^2 \right) \quad dx \right\}^{\frac{1}{2}} . \tag{2.19}$$

Then L_α^λ is a nonnegative self-adjoint operator on $L^2(\Omega, |y|^\lambda d^\alpha(y)dy)$ such that for every $t > 0$, $e^{-L_\alpha^\lambda t}$ has a integral kernel, that is $e^{-L_\alpha^\lambda t} v_0(x) := \int_\Omega l_\alpha^\lambda(t,x,y) v_0(y) |y|^\lambda d^\alpha(y) dy$; here $l_\alpha^\lambda(t,x,y)$ is called the heat kernel of L_α^λ. The existence of $l_\alpha^\lambda(t,x,y)$ can be proved arguing as in [DS1]; that is, using a global Sobolev inequality on Ω, which can be easily deduced from its local version (2.12) as well as (2.18), by means of the partition of unity as in [K].

Then, from the parabolic Harnack inequality in Theorem 2.10, the following sharp two-sided heat kernel estimate for small time, can be easily deduced:

Theorem 2.14 *Let $\alpha \geq 1$, $N \geq 2$, $\lambda \in [2-N, 0]$ and $\Omega \subset \mathbf{R}^N$ be a smooth bounded domain containing the origin. Then there exist positive constants C_1, C_2, with $C_1 \leq C_2$, and $T > 0$ depending on Ω such that*

$$C_1 \min\left\{ \frac{1}{t^{\frac{\alpha}{2}}}, \frac{(|x| + \sqrt{t})^{\frac{|\lambda|}{2}} (|y| + \sqrt{t})^{\frac{|\lambda|}{2}}}{d^{\frac{\alpha}{2}}(x) d^{\frac{\alpha}{2}}(y)} \right\} t^{-\frac{N}{2}} e^{-C_2 \frac{|x-y|^2}{t}} \leq l_\alpha^\lambda(t,x,y) \leq$$

$$\leq C_2 \min\left\{ \frac{1}{t^{\frac{\alpha}{2}}}, \frac{(|x| + \sqrt{t})^{\frac{|\lambda|}{2}} (|y| + \sqrt{t})^{\frac{|\lambda|}{2}}}{d^{\frac{\alpha}{2}}(x) d^{\frac{\alpha}{2}}(y)} \right\} t^{-\frac{N}{2}} e^{-C_1 \frac{|x-y|^2}{t}}$$

for all $x, y \in \Omega$ and $0 < t \leq T$.

Proof of Theorem 2.14: Using the mean value estimate for subsolutions as in Theorem 2.12 and the parabolic Harnack inequality of Theorem 2.10 and arguing as in Theorems 5.2.10, 5.4.10 and 5.4.11 in [SC2] we are lead to the following Li-Yau type estimate

$$\frac{C_1\ e^{-C_2 \frac{|x-y|^2}{t}}}{V(x,\sqrt{t})^{\frac{1}{2}} V(y,\sqrt{t})^{\frac{1}{2}}} \leq l_\alpha^\lambda(t,x,y) \leq \frac{C_2\ e^{-C_1 \frac{|x-y|^2}{t}}}{V(x,\sqrt{t})^{\frac{1}{2}} V(y,\sqrt{t})^{\frac{1}{2}}} ,$$

for all $x, y \in \Omega$ and $0 < t \leq T$; where C_1, C_2 are two positive constants with $C_1 \leq C_2$, and $T > 0$ depends on Ω. From this the result follows using the volume estimate in Lemma 2.2.

//

Using the machinery we have produced in this section we can handle more general operators than the one in Theorems 2.10 and 2.14. Thus, consider the operator

$$\widetilde{L_\alpha^\lambda} := -\frac{1}{|x|^\lambda d^\alpha(x)} \sum_{i,j=1}^N \frac{\partial}{\partial x_i} \left(a_{i,j}(x) |x|^\lambda d^\alpha(x) \frac{\partial}{\partial x_j} \right) , \tag{2.20}$$

where $(a_{i,j}(x))_{N\times N}$ is a measurable symmetric uniformly elliptic matrix. The operator $\widetilde{L_\alpha^\lambda}$ is defined for $\alpha \geq 1$ and $\lambda \in [2-N, 0]$ in $L^2(\Omega, |x|^\lambda d^\alpha(x)\ dx)$ as the generator of the symmetric form

$$\widetilde{\mathcal{L}_\alpha^\lambda}[v_1, v_2] := \sum_{i,j=1}^N \int_\Omega |x|^\lambda d^\alpha(x) a_{i,j}(x) \frac{\partial v}{\partial x_i} \frac{\partial v}{\partial x_j} dx .$$

Then the existence of a heat kernel $\widetilde{l_\alpha^\lambda}(t,x,y)$ follows as in [DS1], and we have

Theorem 2.15 *Let $\alpha \geq 1$, $N \geq 2$, $\lambda \in [2-N, 0]$ and $\Omega \subset \mathbf{R}^N$ be a smooth bounded domain containing the origin. Then there exist positive constants C_1, C_2, with $C_1 \leq C_2$, and $T > 0$ depending on Ω such that*

$$C_1 \min\left\{\frac{1}{t^{\frac{\alpha}{2}}}, \frac{(|x|+\sqrt{t})^{\frac{|\lambda|}{2}}(|y|+\sqrt{t})^{\frac{|\lambda|}{2}}}{d^{\frac{\alpha}{2}}(x)d^{\frac{\alpha}{2}}(y)}\right\} t^{-\frac{N}{2}} e^{-C_2\frac{|x-y|^2}{t}} \leq \widetilde{l_\alpha^\lambda}(t,x,y) \leq$$

$$\leq C_2 \min\left\{\frac{1}{t^{\frac{\alpha}{2}}}, \frac{(|x|+\sqrt{t})^{\frac{|\lambda|}{2}}(|y|+\sqrt{t})^{\frac{|\lambda|}{2}}}{d^{\frac{\alpha}{2}}(x)d^{\frac{\alpha}{2}}(y)}\right\} t^{-\frac{N}{2}} e^{-C_1\frac{|x-y|^2}{t}}$$

for all $x, y \in \Omega$ and $0 < t \leq T$.

Remark 2.16 *A parabolic Harnack inequality up to the boundary similar to the one of Theorem 2.10 can be stated under the same assumptions of Theorem 2.15 for the more general operator $\widetilde{L_\alpha^\lambda}$.*

3 Critical point singularity

In this section we establish a new Improved Hardy inequality (Theorem 3.2) and then we give the proofs of Theorem 1.1 and Theorem 1.2. The structure of this section is as follows.

In Subsection 3.1 we first deduce the improved Hardy inequality and then the global in time pointwise upper bound for the heal kernel of the Schrödinger operator $-\Delta - ((N-2)^2/4)|x|^{-2}$, which is sharp when x and y are close to the boundary (see Theorem 3.4); then, due to an argument contained in [D1], we complete the proof of Theorem 1.2 proving the sharp lower bound for time large enough.

The proof of Theorem 1.1 is finally completed in Subsection 3.2, using the parabolic Harnack inequality up to the boundary of Theorem 2.10.

3.1 Boundary upper bounds and complete sharp description of the heat kernel for large values of time

We first recall the following improved Hardy-Sobolev inequality stated in Theorem A in [FT] (see also inequality (3.3) in [BFT2])

Proposition 3.1 *For $N \geq 3$, let $\Omega \subset \mathbf{R}^N$ be a smooth bounded domain containing the origin and $D \geq \sup_{x\in\Omega}|x|$. Then there exists a positive constant C such that*

$$\int_\Omega |\nabla v|^2 |x|^{2-N} dx \geq C \left(\int_\Omega v^{\frac{2N}{N-2}} |x|^{-N} X^{\frac{2(N-1)}{N-2}}\left(\frac{|x|}{D}\right) dx\right)^{\frac{N-2}{N}} ,$$

for any $v \in C_0^\infty(\Omega)$; here $X(t) = \frac{1}{1-\ln t}$, $t \in (0,1]$.

We next state a new result, the proof of which will be given later on.

Theorem 3.2 (Improved Hardy inequality) *Let $\Omega \subset \mathbf{R}^N$, $N \geq 3$, be a smooth bounded domain containing the origin. Then there exists a constant $C = C(\Omega) \in (0, \frac{1}{4}]$ such that*

$$\int_\Omega \left(|\nabla u|^2 - \frac{(N-2)^2}{4|x|^2}u^2\right) dx \geq C(\Omega) \int_\Omega \frac{u^2}{d^2(x)}\ dx\ , \quad \forall\ u \in C_0^\infty(\Omega)\ . \tag{3.1}$$

The positive constant $C(\Omega)$ can be taken to be exactly $\frac{1}{4}$ for all domains satisfying the following condition

$$-div(|x|^{2-N}\nabla d(x)) \geq 0 \quad a.e.\ in\ \Omega\ . \tag{3.2}$$

For example when $\Omega \equiv B(0, R)$, for arbitrary $R > 0$, condition (3.2) is satisfied. Consequently, in this case the Hardy inequality involving the Schrödinger operator having critical singularity at the origin can be improved exactly by the inverse-square potential having critical singularity at the boundary.

As a consequence of Proposition 3.1 and of the improved Hardy inequality of Theorem 3.2, the following logarithmic Sobolev inequality can be easily obtained:

Theorem 3.3 (Logarithmic Hardy Sobolev inequality) *For $N \geq 3$, let $\Omega \subset \mathbf{R}^N$ be a smooth bounded domain containing the origin. Then for any $u \in C_0^\infty(\Omega \setminus \{0\})$, $u \geq 0$, and any $\epsilon > 0$ we have*

$$\int_\Omega u^2 \log\left(\frac{u}{||u||_2 |x|^{\frac{2-N}{2}} d(x)}\right) dx \leq \epsilon \int_\Omega \left(|\nabla u|^2 - \frac{(N-2)^2}{4|x|^2} u^2\right) dx + \left(K_3 - \frac{N+2}{4}\log\epsilon\right) ||u||_2^2 \ ; \quad (3.3)$$

here K_3 is a positive constant independent of ϵ and $||u||_2 := \left(\int_\Omega u^2 dx\right)^{\frac{1}{2}}$.

Then using Gross theorem of logarithmic Sobolev inequalities, as adapted by Davies and Simon (see Theorem 2.2.7 in [D4]), we will show the following global in time pointwise upper bound for the heat kernel:

Theorem 3.4 *For $N \geq 3$, let $\Omega \subset \mathbf{R}^N$ be a smooth bounded domain containing the origin. Then there exists a positive constant C such that*

$$k(t, x, y) \leq C \frac{d(x) d(y)}{t} (|x||y|)^{\frac{2-N}{2}} t^{-\frac{N}{2}} e^{-\lambda_1 t}, \quad \forall\ x, y \in \Omega,\ t > 0\ .$$

Let us first prove the logarithmic Hardy Sobolev inequality (3.3).

Proof of Theorem 3.3: As a first step we claim that the following logarithmic Hardy Sobolev inequality holds true:

$$\int_\Omega u^2(-\log d(x))\ dx \leq \epsilon \int_\Omega \left(|\nabla u|^2 - \frac{(N-2)^2}{4|x|^2} u^2\right) dx + \left(K_1 - \frac{1}{2}\log\epsilon\right) ||u||_2^2\ , \quad (3.4)$$

for any $u \in C_0^\infty(\Omega)$, $u \geq 0$, and any $\epsilon > 0$; here K_1 is a positive constant independent of ϵ.

To see this let us first suppose that the nonnegative function $u \in C_0^\infty(\Omega)$ is such that $||u||_2 = 1$. We then have

$$\int_\Omega u^2(-\log d(x))\ dx = \frac{1}{2}\int_\Omega u^2(\log d(x)^{-2})\ dx \leq \frac{1}{2}\log\left(\int_\Omega \frac{1}{d(x)^2} u^2 dx\right) \leq$$
$$\leq \frac{1}{2}\log\left(C^{-1}\int_\Omega \left(|\nabla u|^2 - \frac{(N-2)^2}{4|x|^2} u^2\right) dx\right)\ ;$$

here we have used first Jensen's inequality and then the improved Hardy inequality (3.1). For a general nonnegative $u \in C_0^\infty(\Omega)$ we apply the above inequality to the function $\frac{u}{||u||_2}$, to get

$$\int_\Omega u^2(-\log d(x))\ dx \leq \frac{1}{2}||u||_2^2 \log\left(\frac{C^{-1}}{||u||_2^2}\int_\Omega \left(|\nabla u|^2 - \frac{(N-2)^2}{4|x|^2} u^2\right) dx\right)\ .$$

Since $\log z \leq z$ for any $z > 0$, then also $\log y \leq \epsilon 2Cy - \log(\epsilon 2C)$, for any $\epsilon > 0$; whence from this we deduce (3.4), with $K_1 := \frac{1}{2}\log(\frac{1}{2C})$.

We will next show the following logarithmic Hardy Sobolev inequality:

$$\int_\Omega u^2 \log\left(\frac{u}{||u||_2 |x|^{\frac{2-N}{2}}}\right) dx \le \epsilon \int_\Omega \left(|\nabla u|^2 - \frac{(N-2)^2}{4|x|^2} u^2\right) dx + \left(K_2 - \frac{N}{4}\log\epsilon\right) ||u||_2^2 \ , \tag{3.5}$$

for any $u \in C_0^\infty(\Omega \setminus \{0\})$, $u \ge 0$, and any $\epsilon > 0$; here K_2 is a positive constant independent of ϵ.

By Proposition 3.1 it follows easily that there exists a positive constant C such that

$$\int_\Omega |\nabla v|^2 |x|^{2-N} dx \ge C \left(\int_\Omega v^{\frac{2N}{N-2}} |x|^{2-N} dx\right)^{\frac{N-2}{N}} \ , \tag{3.6}$$

for any $v \in C_0^\infty(\Omega)$ (this is inequality (4.12) in [BFT2]). Whence we claim that the following logarithmic Sobolev inequality holds true:

$$\int_\Omega v^2 \log\left(\frac{v}{||v||_2}\right) \ |x|^{2-N} \ dx \le \epsilon \int_\Omega |\nabla v|^2 |x|^{2-N} dx + \left(K_2 - \frac{N}{4}\log\epsilon\right) ||v||_2^2 \ , \tag{3.7}$$

for any $v \in C_0^\infty(\Omega)$, $v \ge 0$, and any $\epsilon > 0$; here K_2 is a positive constant independent of ϵ and $||v||_2 := \left(\int_\Omega v^2 |x|^{2-N} dx\right)^{\frac{1}{2}}$. To see this let us first suppose that the nonnegative function $v \in C_0^\infty(\Omega)$ is such that $||v||_2 = 1$. We then have

$$\int_\Omega v^2 \log(v) \ |x|^{2-N} \ dx = \frac{N-2}{4} \int_\Omega v^2 \log\left(v^{\frac{4}{N-2}}\right) \ |x|^{2-N} \ dx \le \frac{N-2}{4} \log\left(\int_\Omega v^{\frac{4}{N-2}+2} \ |x|^{2-N} \ dx\right) =$$

$$= \frac{N}{4} \log\left(\int_\Omega v^{\frac{2N}{N-2}} \ |x|^{2-N} \ dx\right)^{\frac{N-2}{N}} \le \frac{N}{4} \log\left(C^{-1} \int_\Omega |\nabla v|^2 |x|^{2-N} dx\right) \ ;$$

here we have used first Jensen's inequality and then the improved Hardy-Sobolev inequality (3.6). For a general nonnegative $v \in C_0^\infty(\Omega)$ we apply the above inequality to the function $\frac{v}{||v||_2}$, to get

$$\int_\Omega v^2 \log\left(\frac{v}{||v||_2}\right) \ |x|^{2-N} \ dx \le \frac{N}{4} ||v||_2^2 \log\left(\frac{C^{-1}}{||v||_2^2} \int_\Omega |\nabla v|^2 |x|^{2-N} dx\right) \ .$$

Since $\log z \le z$ for any $z > 0$, then also $\log y \le \epsilon \frac{4C}{N} y - \log\left(\epsilon \frac{4C}{N}\right)$, for any $\epsilon > 0$; whence from this we deduce (3.7) with $K_2 := \frac{N}{4} \log\left(\frac{N}{4C}\right)$.

Inequality (3.7) implies (3.5) via the following change of variables $u := v|x|^{\frac{2-N}{2}}$. Finally from (3.4) and (3.5), the logarithmic Hardy Sobolev inequality (3.3) easily follows with constant $K_3 := K_1 + K_2 + \frac{N+2}{4} \log 2$.

//

We are now ready to give the proof of Theorem 3.4.

Proof of Theorem 3.4: Let us define, as in Section 2 of [D2], the operator $\tilde{K} := U^{-1}(K - \lambda_1)U$, $U : L^2(\Omega, \varphi_1^2 dx) \to L^2(\Omega)$ being the unitary operator $Uw := \varphi_1 w$, thus $\tilde{K} := -\frac{1}{\varphi_1^2} div(\varphi_1^2 \nabla)$. Here $\varphi_1 > 0$ denotes the first eigenfunction and $\lambda_1 > 0$ the first eigenvalue corresponding to the Dirichlet problem $-\Delta\varphi_1 - \frac{(N-2)^2}{4|x|^2}\varphi_1 = \lambda_1 \varphi_1$ in Ω, $\varphi_1 = 0$ on $\partial\Omega$, normalized in such a way that $\int_\Omega \varphi_1^2(x) \ dx = 1$. Due to the results in Lemma 7 in [DD] and using Theorem 7.1 in [DS1] on one hand and elliptic regularity on the other, there exist two positive constants c_1, c_2 such that

$$c_1 |x|^{\frac{2-N}{2}} d(x) \le \varphi_1(x) \le c_2 |x|^{\frac{2-N}{2}} d(x), \quad \forall \ x \in \Omega \ . \tag{3.8}$$

From this and (3.3) we deduce the following logarithmic Sobolev inequality

$$\int_\Omega w^2 \log\left(\frac{w}{||w||_2}\right) \varphi_1^2 dx \le \epsilon < \tilde{K}w, w >_{L^2(\Omega,\varphi_1^2 dx)} + \left(K_4 - \frac{N+2}{4}\log\epsilon\right) ||w||_2^2 \ , \tag{3.9}$$

for any $w \in C_0^\infty(\Omega \setminus \{0\})$, $w \ge 0$, and any $\epsilon > 0$; where $K_4 := K_3 + \lambda_1 - \log c_1$ and $||w||_2 := \left(\int_\Omega w^2\varphi_1^2 \ dx\right)^{\frac{1}{2}}$. Let us remark that only the lower bound in estimate (3.8) was used.

From now on one can use the standard approach of [D4] to complete the proof of the theorem. Here are some details for the convenience of the reader.

As a first step we claim that the following L^p logarithmic Sobolev inequalities holds true:

$$\frac{p}{2}\int_\Omega w^p \log\left(\frac{w}{||w||_p}\right) \varphi_1^2 \ dx \le \epsilon \ \frac{p}{2} < \tilde{K}w, w^{p-1} >_{L^2(\Omega,\varphi_1^2 \ dx)} + \left(K_4 - \frac{N+2}{4}\log\epsilon\right) ||w||_p^p \tag{3.10}$$

for any $w \in C_0^\infty(\Omega \setminus \{0\})$, $w \ge 0$, and any $\epsilon > 0$, $p > 2$. To see this we apply inequality (3.9) to $w^{\frac{p}{2}}$; whence due to the fact that

$$\int_\Omega |\nabla w^{\frac{p}{2}}|^2\varphi_1^2 \ dx = \frac{p^2}{4}\int_\Omega w^{p-2}|\nabla w|^2\varphi_1^2 \ dx = \frac{p^2}{4(p-1)} < \nabla w, \nabla w^{p-1} >_{L^2(\Omega,\varphi_1^2 \ dx)} \le \frac{p}{2} < \tilde{K}w, w^{p-1} >_{L^2(\Omega,\varphi_1^2 \ dx)} ,$$

since $\frac{p}{2(p-1)} \le 1$ if $p \ge 2$; the claim follows.

Let $H_0^1(\Omega, \varphi_1^2 \ dx)$ be the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$||w||_{H^1_{0,\varphi_1^2}} := \left\{\int_\Omega \left(|\nabla w|^2 \ \varphi_1^2 + w^2\varphi_1^2\right) dx\right\}^{\frac{1}{2}} \ ;$$

as one can easily prove this is also the closure of $C_0^\infty(\Omega \setminus \{0\})$ with respect to the same norm. Then to the operator $\tilde{K}$ defined in the domain $D(\tilde{K}) = \{w \in H_0^1(\Omega, \varphi_1^2 \ dx) : \tilde{K}w \in L^2(\Omega, \varphi_1^2 \ dx)\}$ it is naturally associated the bilinear symmetric form defined as follows $\tilde{\mathcal{K}}[w_1, w_2] :=< \tilde{K}w_1, w_2 >_{L^2(\Omega,\varphi_1^2 \ dx)}= \int_\Omega \nabla w_1 \nabla w_2 \ \varphi_1^2 \ dx$, which is a Dirichlet form. Whence Lemma 1.3.4 and Theorems 1.3.2 and 1.3.3 in [D4] implies that $e^{-\tilde{K}t}$, which is an analytic contraction semigroup in $L^2(\Omega, \varphi_1^2 \ dx)$, is also positivity preserving and a contraction semigroup in $L^p(\Omega, \varphi_1^2 \ dx)$ for any $1 \le p \le \infty$. As a consequence for any $t > 0$ and any $p \ge 2$

$$e^{-\tilde{K}t}[L^2(\Omega, \varphi_1^2 \ dx) \cap L^\infty(\Omega)]^+ \subset [H_0^1(\Omega, \varphi_1^2 \ dx) \cap L^p(\Omega, \varphi_1^2 \ dx) \cap L^\infty(\Omega)]^+ \ ;$$

where we denote by $[E]^+$ the subset of positive functions in the space E.

Thus by density argument the L^p logarithmic Sobolev inequality (3.10), more generally applies to any function in $\cup_{t>0} \ e^{-\tilde{K}t}[L^2(\Omega, \varphi_1^2 dx) \cap L^\infty(\Omega)]^+$.

This means that Theorem 2.2.7 in [D4] can be applied, in the same way as in Corollary 2.2.8 in [D4], to the operator $\tilde{K}$; whence obtaining that

$$||e^{-\tilde{K}t}||_{2\to\infty} \le Ct^{-\frac{N+2}{4}} \ ,$$

and by duality that

$$||e^{-\tilde{K}t}||_{1\to 2} \le Ct^{-\frac{N+2}{4}} \ ,$$

that is

$$||e^{-\tilde{K}t}||_{1\to\infty} \le Ct^{-\frac{N+2}{2}} \ .$$

Here we use the following notation:

$$||e^{-\tilde{K}t}||_{q\to p} := \sup_{0<||f||_q\leq 1} \frac{||e^{-\tilde{K}t} f(x)||_p}{||f(x)||_q} \ ,$$

where $||f||_q := \left(\int_\Omega |f|^q \varphi_1^2 \ dx\right)^{\frac{1}{q}}$. This implies, by Dunford-Pettis theorem, that the semigroup $e^{-\tilde{K}t}$ is indeed a semigroup of integral operators; that is a heat kernel $\tilde{k}(t,x,y)$ associated to the semigroup $e^{-\tilde{K}t}$ is well defined and satisfies the following pointwise upper bound $\tilde{k}(t,x,y) \leq C\frac{1}{t}t^{-\frac{N}{2}}$, for any $x,y \in \Omega$ and any $t > 0$. Theorem 3.4 then follows, due to the upper bound in (3.8) and to the fact that, as a consequence of the unitary operator U, the heat kernels $k(t,x,y)$ and $\tilde{k}(t,x,y)$, corresponding respectively to K and $\tilde{K}$, satisfy the following equivalence

$$k(t,x,y) \equiv \varphi_1(x)\varphi_1(y) \ \tilde{k}(t,x,y) e^{-\lambda_1 t} \ . \tag{3.11}$$

//

Remark 3.5 *Applying Davies's method of exponential perturbation to the operator $\tilde{K}$ (see Section 2 in [D3] for details), the upper bound in Theorem 3.4 can be improved by adding a factor $c_\delta e^{-\frac{|x-y|^2}{4(1+\delta)t}}$.*

Let us now deduce from the upper bound in Theorem 3.4 an analogous lower bound for time large enough, thus completing the proof of Theorem 1.2. We argue as in Theorem 6 of [D1] (see also Proposition 4 of [D2]), we give the details here for the convenience of the reader.

Proof of Theorem 1.2: Making use of the same notation as in the proof of Theorem 3.4, the lower bound we want to prove corresponds to the statement $\tilde{k}(t,x,y) \geq C$ for any $x,y \in \Omega$ if t is large enough, C being some positive constant.

For any $f \in L^1(\Omega, \varphi_1^2 \ dx)$, we clearly have

$$f =< f, 1 > 1 + g,$$

where $< f,1 >:=< f,1 >_{L^2(\Omega,\varphi_1^2 \ dx)}$, and $< g,1 >= 0$, since $\int_\Omega \varphi_1^2(x)dx = 1$. Thus, making use of the fact that by definition $\tilde{K}1 = 0$ we have

$$e^{-\tilde{K}t} f =< f,1 > 1 + e^{-\tilde{K}t} g,$$

that is the semigroup $e^{-At}f := e^{-\tilde{K}t}f- < f,1 > 1$, to whom it is clearly associated the heat kernel $\tilde{k}(t,x,y) - 1$, is such that for any $f \in L^1(\Omega, \varphi_1^2 \ dx)$

$$e^{-At} f \equiv e^{-\tilde{K}t} g,$$

where $g = g(f)$ is a function in $L^1(\Omega, \varphi_1^2 dx)$ such that $< g,1 >= 0$. Thus, due to Theorem 3.4

$$||e^{-At}||_{1\to\infty} \leq ||e^{-\tilde{K}t}||_{1\to\infty} \leq Ct^{-\frac{N+2}{2}},$$

here C is some positive constant; this is equivalent to say that

$$|\tilde{k}(t,x,y) - 1| \leq Ct^{-\frac{N+2}{2}} \ ,$$

from which the claim easily follows for t large enough.

//

In the sequel we will give the proof of Theorem 3.2. We will use the following lemma whose proof will be postponed until the end of this subsection.

Lemma 3.6 *For $N \geq 3$, let $\Omega \subset \mathbf{R}^N$ be a smooth bounded domain containing the origin. Then there exists $\delta_0 > 0$, such that*

$$\inf_{f \in C_0^\infty(\Omega_\delta)} \frac{\int_{\Omega_\delta} |x|^{2-N} |\nabla f|^2 dx}{\int_{\Omega_\delta} |x|^{2-N} \frac{f^2}{d^2(x)} dx} = \frac{1}{4} \ ,$$

for all $0 < \delta \leq \delta_0$; here $\Omega_\delta := \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) \leq \delta\}$.

We are now ready to prove the improved Hardy inequality.

Proof of Theorem 3.2: (i) Let us first prove the claim on any domain Ω satisfying condition (3.2). To this end let us define for any $u \in C_0^\infty(\Omega)$ as a new variable $w := |x|^{\frac{N-2}{2}} d^{-\frac{1}{2}}(x) u$, obviously $w \in H_0^1(\Omega)$. By direct computations we have

$$\nabla u = \frac{2-N}{2} |x|^{-\frac{N}{2}-1} \ x \ d^{\frac{1}{2}} \ w + |x|^{\frac{2-N}{2}} \frac{1}{2} d^{-\frac{1}{2}} \nabla d \ w \ + |x|^{\frac{2-N}{2}} d^{\frac{1}{2}} \nabla w$$

thus

$$\int_\Omega |\nabla u|^2 = \int_\Omega \left(\frac{(N-2)^2}{4} |x|^{-N} \ d \ w^2 + |x|^{2-N} \frac{1}{4} d^{-1} \ w^2 \ + |x|^{2-N} d \ |\nabla w|^2 \right) dx +$$

$$+ \int_\Omega \left(-\frac{N-2}{2} |x|^{-N} \ w^2 \ x \ \nabla d - (N-2) |x|^{-N} \ d \ w \ x \ \nabla w + \nabla d \nabla w \ w \ |x|^{2-N} \right) dx \ .$$

Whence

$$\int_\Omega \left(|\nabla u|^2 - \frac{(N-2)^2}{4|x|^2} u^2 - \frac{1}{4d^2} u^2 \right) dx =$$

$$= \int_\Omega \left(|\nabla w|^2 \ d \ |x|^{2-N} - \frac{N-2}{2} |x|^{-N} \ w^2 \ x \ \nabla d - \frac{(N-2)}{2} |x|^{-N} \ d \ x \ \nabla w^2 + \frac{1}{2} \ \nabla d \ \nabla w^2 \ |x|^{2-N} \right) dx =$$

$$= \int_\Omega \left(|\nabla w|^2 d |x|^{2-N} - \frac{N-2}{2} |x|^{-N} \ w^2 \ x \ \nabla d + \frac{(N-2)}{2} div(|x|^{-N} d \ x) w^2 - \frac{1}{2} div(|x|^{2-N} \nabla d) w^2 \right) dx =$$

$$= \int_\Omega \left(|\nabla w|^2 \ d \ |x|^{2-N} - \frac{1}{2} div(|x|^{2-N} \nabla d) w^2 \right) dx \geq 0,$$

due to condition (3.2) on Ω. Thus inequality (3.1) is proved with constant $C(\Omega) \equiv \frac{1}{4}$ in any domain Ω satisfying condition (3.2).

(ii) Let us prove indirectly the claim in the remaining case. To this end let us denote by $H_0^1(\Omega, |x|^{2-N} dx)$ the closure of $C_0^\infty(\Omega)$ in the norm

$$||f||_{H^1_{2-N}} := \left\{ \int_\Omega (|\nabla f|^2 + f^2) |x|^{2-N} dx \right\}^{\frac{1}{2}} \ . \tag{3.12}$$

The improved Hardy inequality (3.1) we are going to prove, in the new variable $v := |x|^{\frac{N-2}{2}} u$ reads as follows

$$\int_\Omega |\nabla v|^2 |x|^{2-N} dx \geq C \int_\Omega |x|^{2-N} \frac{v^2}{d^2} dx \ .$$

Let us suppose that the improved Hardy inequality (3.1) is false; whence let us suppose that the following holds true

$$\inf_{\{\int_\Omega |x|^{2-N} \frac{v^2}{d^2} \ dx \ = \ 1\}} \int_\Omega |x|^{2-N} |\nabla v|^2 dx = 0 \ ;$$

thus there exists a sequence $\{v_j\}_{j\geq 0}$ in $H_0^1(\Omega,|x|^{2-N}dx)$ such that $\int_\Omega |x|^{2-N}\frac{v_j^2}{d^2}\,dx=1$, and

$$\int_\Omega |x|^{2-N}|\nabla v_j|^2dx \to 0, \qquad \text{as } j\to\infty\ . \tag{3.13}$$

For any arbitrary function $\varphi\in C_0^\infty(\Omega)$, such that $\varphi\equiv 1$ in a neighborhood of the origin, we also have

$$\int_\Omega |x|^{2-N}|\nabla(\varphi v_j)|^2dx \leq 2\int_\Omega |x|^{2-N}\left(|\nabla v_j|^2\varphi^2+|\nabla\varphi|^2v_j^2\right)dx \leq$$

$$\leq C\int_\Omega |x|^{2-N}\left(|\nabla v_j|^2+v_j^2\right)dx \leq C\int_\Omega |x|^{2-N}|\nabla v_j|^2dx \to 0 \text{ as } j\to\infty\ . \tag{3.14}$$

Here we use the fact that the following inequality holds true

$$\int_\Omega |x|^{2-N}f^2dx \leq C\int_\Omega |x|^{2-N}|\nabla f|^2dx, \quad \forall\ f\in H_0^1(\Omega,|x|^{2-N}dx)\ . \tag{3.15}$$

Inequality (3.15) for example follows easily from inequality (3.6) by Holder inequality. From estimate (3.14) and inequality (3.15) (applied to $f:=\varphi v_j$) we easily deduce that

$$\int_\Omega |x|^{2-N}\varphi^2v_j^2 \to 0, \qquad \text{as } j\to\infty,$$

or similarly (due to the fact that φ has compact support inside Ω) that

$$\int_\Omega |x|^{2-N}\varphi^2\frac{v_j^2}{d^2}dx \to 0, \qquad \text{as } j\to\infty\ . \tag{3.16}$$

We then compute

$$1=\int_\Omega |x|^{2-N}\frac{v_j^2}{d^2}dx=\int_\Omega |x|^{2-N}\frac{(\varphi v_j+(1-\varphi)v_j)^2}{d^2}dx=$$

$$=\int_\Omega |x|^{2-N}\varphi^2\frac{v_j^2}{d^2}dx+2\int_\Omega |x|^{2-N}\varphi(1-\varphi)\frac{v_j^2}{d^2}dx+\int_\Omega |x|^{2-N}(1-\varphi)^2\frac{v_j^2}{d^2}dx.$$

We observe that the first two terms in the last line tend to zero as j tends to infinity and therefore we obtain that

$$\int_\Omega |x|^{2-N}(1-\varphi)^2\frac{v_j^2}{d^2}dx = 1+o(1), \qquad \text{as } j\to\infty\ . \tag{3.17}$$

On the other hand we have that

$$\int_\Omega |x|^{2-N}|\nabla[(1-\varphi)v_j]|^2dx \leq 2\int_\Omega |x|^{2-N}|\nabla v_j|^2dx+2\int_\Omega |x|^{2-N}|\nabla(\varphi v_j)|^2dx\ ,$$

both terms in the right hand side going to zero as j tends to infinity due to (3.13) and (3.14); whence we deduce that

$$\int_\Omega |x|^{2-N}|\nabla[(1-\varphi)v_j]|^2dx \to 0, \qquad \text{as } j\to\infty\ . \tag{3.18}$$

Since for any $j\geq 0$ the function $f:=(1-\varphi)v_j$ is an element of $H_0^1(\Omega_\delta)$ for a suitable choice of the function φ (take it identically one in a subset containing $\Omega\setminus\Omega_\delta$), by means of (3.17) and (3.18) we reach a contradiction with Lemma 3.6, thus proving the improved Hardy inequality.

//

A similar improved Hardy inequality for a potential behaving like $((N-2)^2/4)|x|^{-2}$ near the origin and exactly like $(1/4)d^{-2}(x)$ near the boundary is also shown without any geometric assumption on the domain Ω (see Theorem 3.10 below).

We next prove Lemma 3.6. One can consider it as a consequence of the following more general result.

Lemma 3.7 *For $N \geq 3$, let $\Omega \subset \mathbf{R}^N$ be a smooth bounded domain. Then there exists a positive constant $\delta_0 = \delta_0(\Omega)$, such that for any $V \in L^{\frac{N}{2}}(\Omega_{\delta_0})$ and $0 < \delta \leq \delta_0$, we have the following estimate*

$$\int_{\Omega_\delta} \left(|\nabla u|^2 - \frac{1}{4d^2} u^2 \right) dx \geq c \int_{\Omega_\delta} V u^2 dx \ , \quad \forall\, u \in C_0^\infty(\Omega_\delta) \ ;$$

here $c = c(\delta) \to \infty$ as $\delta \to 0$ and $\Omega_\delta := \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) \leq \delta\}$.

Proof of Lemma 3.6: Let us choose $V(x) := \frac{(N-2)^2}{4|x|^2}$ in Lemma 3.7 above and let us choose δ small enough such that $c(\delta) \geq 1$ and $0 < \delta \leq \delta_0$, thus we have

$$\int_{\Omega_\delta} \left(|\nabla u|^2 - \frac{1}{4d^2} u^2 \right) dx \geq \int_{\Omega_\delta} \frac{(N-2)^2}{4|x|^2} u^2 dx, \tag{3.19}$$

for any $u \in C_0^\infty(\Omega_\delta)$. For any $f \in C_0^\infty(\Omega_\delta)$, $u := f|x|^{\frac{2-N}{2}}$ will be in $C_0^\infty(\Omega_\delta)$, moreover by easy computations we have

$$\int_{\Omega_\delta} \left(|\nabla u|^2 - \frac{1}{4d^2} u^2 \right) dx = \int_{\Omega_\delta} \left(|\nabla f|^2 + \frac{(N-2)^2}{4|x|^2} f^2 - \frac{1}{4d^2} f^2 \right) |x|^{2-N} dx \ ,$$

thus (3.19) can be restated as follows

$$\int_{\Omega_\delta} \left(|\nabla f|^2 - \frac{1}{4d^2} f^2 \right) |x|^{2-N} dx \geq 0 \ ;$$

this proves the claim.

//

Whence it only remains to prove Lemma 3.7. Before doing so let us observe that inequality (3.19), simply says that the improved Hardy inequality (3.1) indeed holds true with constant $C(\Omega) = \frac{1}{4}$ whenever the support of the functions considered is contained in a neighborhood of the boundary.

The proof of Lemma 3.7 makes use of the following improved Hardy-Sobolev inequality near the boundary stated in Theorem 3 of [FMT1], we recall it here for the convenience of the reader:

Proposition 3.8 *For $N \geq 3$, let $\Omega \subset \mathbf{R}^N$ be a smooth bounded domain. Then there exist positive constants $\delta_0 = \delta_0(\Omega)$ and $C = C(N)$, such that*

$$\int_{\Omega_\delta} \left(|\nabla u|^2 - \frac{1}{4d^2} u^2 \right) dx \geq C \left(\int_{\Omega_\delta} u^{\frac{2N}{N-2}} dx \right)^{\frac{N-2}{N}} \ , \quad \forall\, u \in C_0^\infty(\Omega_\delta),$$

and any $0 < \delta \leq \delta_0$; here $\Omega_\delta := \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) \leq \delta\}$.

Let us focus here on the fact that in Proposition 3.8 no convexity assumption on the domain Ω is made; this is due to the fact that we only consider functions whose supports are contained in a neighborhood of the boundary.

Proof of the Lemma 3.7: By Holder inequality we have

$$\int_{\Omega_\delta} Vu^2 dx \le \left(\int_{\Omega_\delta} V^{\frac{N}{2}} dx\right)^{\frac{2}{N}} \left(\int_{\Omega_\delta} u^{\frac{2N}{N-2}} dx\right)^{\frac{N-2}{N}} \le$$

$$\le \left(\int_{\Omega_\delta} V^{\frac{N}{2}} dx\right)^{\frac{2}{N}} C(N)^{-1} \int_{\Omega_\delta} \left(|\nabla u|^2 - \frac{1}{4}\frac{u^2}{d^2}\right) dx \ ,$$

the last step being due to Proposition 3.8. This proves the claim with constant

$$c(\delta) := \left(\int_{\Omega_\delta} V^{\frac{N}{2}} dx\right)^{-\frac{2}{N}} C(N) \ ,$$

which tends to infinity as δ tends to zero due to the integrability assumption on V.

//

With some minor changes in the proof of Theorem 3.2 one can indeed prove the following improved Hardy inequality, which does not a priori requires the bounded domain Ω to be smooth.

Theorem 3.9 *For $N \ge 3$, let $\Omega \subset \mathbf{R}^N$ be a bounded domain containing the origin such that*

$$\int_\Omega |\nabla u|^2 dx \ge C \int_\Omega \frac{u^2}{d^2} dx, \quad \forall\ u \in C_0^\infty(\Omega)$$

and some positive constant C. Then there exists a positive constant $\tilde{C}$ such that

$$\int_\Omega \left(|\nabla u|^2 - \frac{(N-2)^2}{4|x|^2} u^2\right) dx \ge \tilde{C} \int_\Omega \frac{u^2}{d^2} dx, \quad \forall\ u \in C_0^\infty(\Omega) \ .$$

We finally mention the following related new Hardy inequality, which we think is of independent interest

Theorem 3.10 *For $N \ge 3$, let $\Omega \subset \mathbf{R}^N$ be a smooth bounded domain containing the origin, and define for $\epsilon > 0$,*

$$V_\epsilon(x) = \begin{cases} \frac{(N-2)^2}{4|x|^2} & if\ \{x \in \Omega : d(x) \ge \epsilon\} \\ \frac{1}{4d^2(x)} & if\ \{x \in \Omega : d(x) < \epsilon\} \ . \end{cases}$$

Then there exists $\epsilon_0 = \epsilon_0(\Omega)$ such that for all $0 < \epsilon \le \epsilon_0$ and $u \in C_0^\infty(\Omega)$, we have

$$\int_\Omega |\nabla u|^2 dx \ge \int_\Omega V_\epsilon(x) u^2 dx \ .$$

Proof: We will only sketch it. Let $\Omega_1 = \{x \in \Omega : d(x) \ge \epsilon\}$. Then using the change of variable $u := |x|^{\frac{2-N}{2}} v$, one can prove the following inequality

$$\int_{\Omega_1} \left(|\nabla u|^2 - \frac{(N-2)^2}{4|x|^2} u^2\right) dx \ge \frac{2-N}{2} \int_{\partial\Omega_1} \frac{u^2}{|x|^2} x \cdot \nu \ dS_x.$$

Similarly using the change of variable $u := d^{\frac{1}{2}}(x)X^{-\frac{1}{2}}(d(x))v$ with $X(t) = (1-\ln t)^{-1}$ one can prove the following inequality

$$\int_{\Omega\setminus\Omega_1}\left(|\nabla u|^2 - \frac{1}{4d^2(x)}u^2\right)dx \geq -\frac{1}{4}\int_{\partial\Omega_1}\frac{u^2}{d(x)}\nabla d\cdot\nu\, dS_x,$$

for any $0 < \epsilon \leq \min\{e^{-1}, \epsilon_1\}$ where $\epsilon_1 > 0$ is such that $d^{-1}X(d) + 2\Delta d(\ln d) \geq 0$ for $d \leq \epsilon_1$. The result then follows showing that for $0 < \epsilon \leq \epsilon_0 = \min\{e^{-1}, \epsilon_1, \frac{R}{2N-3}\}$ we have $\left[\frac{2(2-N)}{|x|^2}x - \frac{1}{d(x)}\nabla d\right]\cdot\nu \geq 0$ since $\nu := -\nabla d$ on $\partial\Omega_1$; here R denotes a positive constant such that $B(0,R)\subset\Omega$, which exists due to the assumption on Ω. //

3.2 Complete sharp description of the heat kernel for small values of time

In this section we prove the two-sided sharp estimate on the heat kernel $k(t,x,y)$ stated for small time in Theorem 1.1.

Proof of Theorem 1.1 Since for any $x\in\Omega$ and for some positive constants c_1, c_2 we have the following estimate $c_1|x|^{\frac{\lambda}{2}}d^{\frac{\alpha}{2}}(x) \leq \varphi_1(x) \leq c_2|x|^{\frac{\lambda}{2}}d^{\frac{\alpha}{2}}(x)$ for $\alpha = 2$ and $\lambda = 2-N$, we can apply the result of Theorem 2.15 to the operator $\tilde{K} = -\frac{1}{\varphi_1^2(x)}div(\varphi_1^2(x)\nabla)$. Hence due to (3.11) the result follows immediately.

//

Let us finally make some remarks concerning Schrödinger operators having potential $V(x) = c|x|^{-2}$. Arguing as in Lemma 7 in [DD] one can easily prove that the first Dirichlet eigenfunction for the Schrödinger operator $-\Delta - \frac{c}{|x|^2}$, $0 < c < \frac{(N-2)^2}{4}$, behaves like $|x|^{\frac{\lambda}{2}}d(x)$ on all Ω, where $\lambda := 2-N+\sqrt{(N-2)^2-4c}$. Then we have

Theorem 3.11 *For $N\geq 3$, let $\Omega\subset\mathbf{R}^N$ be a smooth bounded domain containing the origin. Then there exist positive constants C_1, C_2, with $C_1\leq C_2$, and $T>0$ depending on Ω such that*

$$C_1\min\left\{(|x|+\sqrt{t})^{\frac{|\lambda|}{2}}(|y|+\sqrt{t})^{\frac{|\lambda|}{2}}, \frac{d(x)d(y)}{t}\right\}(|x||y|)^{\frac{\lambda}{2}}t^{-\frac{N}{2}}e^{-C_2\frac{|x-y|^2}{t}} \leq$$

$$\leq k_c(t,x,y) \leq C_2\min\left\{(|x|+\sqrt{t})^{\frac{|\lambda|}{2}}(|y|+\sqrt{t})^{\frac{|\lambda|}{2}}, \frac{d(x)d(y)}{t}\right\}(|x||y|)^{\frac{\lambda}{2}}t^{-\frac{N}{2}}e^{-C_1\frac{|x-y|^2}{t}} ,$$

for all $x,y\in\Omega$ and $0<t\leq T$; here $k_c(t,x,y)$ denotes the heat kernel associated to the operator $-\Delta-\frac{c}{|x|^2}$ in Ω under Dirichlet boundary conditions for $0<c<\frac{(N-2)^2}{4}$, and $\lambda := 2-N+\sqrt{(N-2)^2-4c}$.

Theorem 3.12 *For $N\geq 3$, let $\Omega\subset\mathbf{R}^N$ be a smooth bounded domain containing the origin. Then there exist two positive constants C_1, C_2, with $C_1\leq C_2$, such that*

$$C_1\ d(x)\ d(y)\ (|x||y|)^{\frac{\lambda}{2}}e^{-\lambda_1 t} \leq k_c(t,x,y) \leq C_2\ d(x)\ d(y)\ (|x||y|)^{\frac{\lambda}{2}}e^{-\lambda_1 t} ,$$

for all $x,y\in\Omega$ and $t>0$ large enough; here $k_c(t,x,y)$ denotes the heat kernel associated to the operator $-\Delta-\frac{c}{|x|^2}$ in Ω under Dirichlet boundary conditions for $0<c<\frac{(N-2)^2}{4}$, λ_1 its (positive) elliptic first eigenvalue and $\lambda := 2-N+\sqrt{(N-2)^2-4c}$.

4 Critical boundary singularity

In this section we prove Theorems 1.3 and 1.4 as well as a new Hardy-Moser inequality (Theorem 4.3). The structure of this section is as follows.

In Subsection 4.1 we first prove the improved Hardy-Moser inequality. Then in Subsection 4.2 we get the global in time pointwise upper bound for the heal kernel of the Schrödinger operator $-\Delta-(1/4)d^{-2}(x)$, which is sharp when x and y are close to the boundary (see Theorem 4.4). Then arguing as in [D1], we deduce the sharp heat kernel lower bound for time large enough, thus completing the proof of Theorem 1.4.

The proof of Theorem 1.3 is finally completed in Subsection 4.3, using the parabolic Harnack inequality up to the boundary stated in Theorem 1.5.

4.1 The improved Hardy-Moser inequality

Here we will prove a new improved Hardy-Moser inequality which we think it is of independent interest. The proof is based on an auxiliary Hardy-Sobolev inequality, that we will show here, as well as on the following improved Hardy inequality stated in Theorem A in [BFT1].

Proposition 4.1 *For $N \geq 2$, let $\Omega \subset \mathbf{R}^N$ be a smooth bounded and convex domain. Then there exists D_0 positive such that for all $D \geq D_0$*

$$\int_\Omega \left(|\nabla u|^2 - \frac{1}{4d^2(x)} u^2 \right) \, dx \geq \frac{1}{4} \int_\Omega \frac{X^2\left(\frac{d(x)}{D}\right)}{d^2(x)} u^2 \, dx \ ,$$

for any $u \in C_0^\infty(\Omega)$; here $X(t) := \frac{1}{1-\ln t}$, $t \in (0,1]$.

Let us now state the auxiliary Hardy-Sobolev inequality we will use in the sequel.

Lemma 4.2 *Let $\alpha > 0$, $N \geq 2$ and $\Omega \subset \mathbf{R}^N$ be a smooth bounded domain. Then there exist $\delta_0 > 0$ and $C = C(\alpha, \delta_0) > 0$ such that*

$$\int_\Omega d^\alpha(x)|\nabla v| \, dx + \int_{\Omega\setminus\Omega_\delta} d^{\alpha-1}(x)|v| \, dx \geq C \left(\int_\Omega d^{\frac{\alpha N}{N-1}}(x)|v|^{\frac{N}{N-1}} \, dx \right)^{\frac{N-1}{N}} \ ,$$

for any $v \in C_0^\infty(\Omega)$ and any $0 < \delta \leq \delta_0$; here $\Omega_\delta := \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) \leq \delta\}$.

Proof: We will follow closely the argument of [FMT2]. Our starting point is the following Gagliardo-Nirenberg inequality (see p. 189 in [M])

$$S_N ||f||_{L^{\frac{N}{N-1}}(\Omega)} \leq ||\nabla f||_{L^1(\Omega)} \ , \quad \forall \ f \in C_0^\infty(\Omega) \ ,$$

where S_N is a positive constant depending only on N.

For any $v \in C_0^\infty(\Omega)$ let us apply the above inequality to the function $f := d^\alpha v$. Hence we obtain

$$S_N ||d^\alpha v||_{L^{\frac{N}{N-1}}(\Omega)} \leq \int_\Omega d^\alpha(x)|\nabla v| dx + \alpha \int_\Omega d^{\alpha-1}(x)|v| dx \ .$$

We next estimate the last term above. Let $\varphi_\delta \in C_0^\infty(\Omega_{2\delta})$, $0 \leq \varphi_\delta \leq 1$, be a cut off function which is identically one in Ω_δ and identically zero in $\mathbf{R}^N \setminus \Omega_{2\delta}$. Clearly $v = \varphi_\delta v + (1-\varphi_\delta)v$. Then we have

$$\alpha \int_\Omega d^{\alpha-1}(x)|v| dx \leq \alpha \int_\Omega d^{\alpha-1}(x)|\varphi_\delta v| dx + \alpha \int_\Omega d^{\alpha-1}(x)(1-\varphi_\delta)|v| dx \leq$$

$$\leq \alpha \int_\Omega d^{\alpha-1}(x)|\varphi_\delta v|dx + \alpha \int_{\Omega\setminus\Omega_\delta} d^{\alpha-1}(x)|v|dx \ .$$

Concerning the first term on the right hand side we have

$$\alpha \int_\Omega d^{\alpha-1}(x)|\varphi_\delta v|dx = \int_\Omega \nabla d^\alpha \cdot \nabla d|\varphi_\delta v|dx = -\int_\Omega d^\alpha(x)\nabla d\cdot\nabla|\varphi_\delta v|dx - \int_\Omega d^\alpha(x)\Delta d|\varphi_\delta v|dx \leq$$

$$\leq \int_\Omega d^\alpha(x)|\nabla(\varphi_\delta v)|dx + c_0\delta\int_\Omega d^{\alpha-1}(x)|\varphi_\delta v|dx \ ,$$

here we used the smoothness assumption on Ω which implies that $|d\Delta d| \leq c_0\delta$ in Ω_δ for δ small, say $0<\delta\leq\delta_0$, and for some positive constant c_0 independent of δ (δ_0, c_0 depending on Ω). Thus we have for any $0<\delta\leq\delta_0$

$$\alpha\int_\Omega d^{\alpha-1}(x)|\varphi_\delta v|dx \leq \frac{\alpha}{\alpha-c_0\delta_0}\int_\Omega d^\alpha(x)|\nabla(\varphi_\delta v)|dx \leq C\int_\Omega d^\alpha(x)|\nabla v|\varphi_\delta dx +$$

$$+\frac{C}{\delta}\int_{\Omega_{2\delta}\setminus\Omega_\delta} d^\alpha(x)|v|dx \leq C\int_\Omega d^\alpha(x)|\nabla v|dx + C\int_{\Omega_{2\delta}\setminus\Omega_\delta} d^{\alpha-1}(x)|v|dx \ ,$$

from which the result follows.

//

We next state the new improved Hardy-Moser inequality.

Theorem 4.3 (Improved Hardy-Moser inequality) *For $N\geq 2$, let $\Omega\subset\mathbf{R}^N$ be a smooth bounded and convex domain. Then there exists a positive constant C such that*

$$\left\{\int_\Omega\left(|\nabla u|^2-\frac{1}{4d^2}u^2\right)\ dx\right\}\left(\int_\Omega u^2dx\right)^{\frac{2}{N}} \geq C\int_\Omega u^{2(1+\frac{2}{N})}dx \ , \quad \forall\ u\in C_0^\infty(\Omega) \ .$$

Proof: Changing variables by $v := ud^{-\frac{1}{2}}$, we get

$$\int_\Omega u^{\frac{2(N+2)}{N}}dx = \int_\Omega d^{\frac{N+2}{N}}v^{\frac{2(N+2)}{N}}dx = \int_\Omega d^{\frac{\alpha N}{N-1}}(v^{2\alpha})^{\frac{N}{N-1}}dx \ ,$$

with $\alpha := \frac{(N+2)(N-1)}{N^2}$. Applying Lemma 4.2 to the function $v^{2\alpha}$ we have

$$\int_\Omega d^{\frac{\alpha N}{N-1}}(v^{2\alpha})^{\frac{N}{N-1}}dx \leq C\left(\int_\Omega d^\alpha|\nabla v^{2\alpha}|dx + \int_{\Omega\setminus\Omega_\delta} d^{\alpha-1}v^{2\alpha}dx\right)^{\frac{N}{N-1}} \leq$$

$$\leq C\left(2\alpha\int_\Omega d^\alpha|\nabla v||v|^{2\alpha-1}dx + \int_{\Omega\setminus\Omega_\delta} d^{\alpha-1}v^{2\alpha}dx\right)^{\frac{N}{N-1}} \leq$$

$$\leq C\left\{\left(\int_\Omega d(x)|\nabla v|^2\ dx\right)^{\frac{1}{2}}\left(\int_\Omega d^{2\alpha-1}v^{2(2\alpha-1)}dx\right)^{\frac{1}{2}} + \left(\int_{\Omega\setminus\Omega_\delta}\frac{v^2}{d}dx\right)^{\frac{1}{2}}\left(\int_{\Omega\setminus\Omega_\delta} d^{2\alpha-1}v^{2(2\alpha-1)}dx\right)^{\frac{1}{2}}\right\}^{\frac{N}{N-1}} \leq$$

$$\leq C\left(\int_\Omega d^{2\alpha-1}v^{2(2\alpha-1)}dx\right)^{\frac{N}{2(N-1)}}\left\{\int_\Omega\left(|\nabla v|^2 d - \frac{1}{2}\Delta d\ v^2\right)dx\right\}^{\frac{N}{2(N-1)}} \ ;$$

here we used Proposition 4.1 (observe that $\frac{1}{4}X^2(\frac{d}{D}) \geq \frac{1}{4}X^2(\frac{\delta}{D})$ if $x \in \Omega \setminus \Omega_\delta$) and standard estimates. Returning to the original variable u, we obtain

$$\int_\Omega u^{\frac{2(N+2)}{N}} dx \leq C \left(\int_\Omega u^{2(2\alpha-1)} dx \right)^{\frac{N}{2(N-1)}} \left\{ \int_\Omega \left(|\nabla u|^2 - \frac{1}{4d^2} u^2 \right) dx \right\}^{\frac{N}{2(N-1)}} ,$$

that is,

$$\left(\int_\Omega u^{\frac{2(N+2)}{N}} dx \right)^{\frac{2(N-1)}{N}} \leq C \left(\int_\Omega u^{2(2\alpha-1)} dx \right) \left\{ \int_\Omega \left(|\nabla u|^2 - \frac{1}{4d^2} u^2 \right) dx \right\} .$$

If $N = 2$ we have that $\alpha = 1$, thus the above inequality becomes

$$\int_\Omega u^4 dx \leq C \left(\int_\Omega u^2 dx \right) \left\{ \int_\Omega \left(|\nabla u|^2 - \frac{1}{4d^2} u^2 \right) dx \right\}$$

which is the sought for estimate. For $N \geq 3$, we use Hölder inequality to obtain

$$\left(\int_\Omega u^{\frac{2(N+2)}{N}} dx \right)^{\frac{2(N-1)}{N}} \leq C \left(\int_\Omega u^2 dx \right)^{\frac{2}{N}} \left(\int_\Omega u^{\frac{2(N+2)}{N}} dx \right)^{\frac{N-2}{N}} \left\{ \int_\Omega \left(|\nabla u|^2 - \frac{1}{4d^2} u^2 \right) dx \right\}$$

from which

$$\int_\Omega u^{\frac{2(N+2)}{N}} dx \leq C \left(\int_\Omega u^2 dx \right)^{\frac{2}{N}} \left\{ \int_\Omega \left(|\nabla u|^2 - \frac{1}{4d^2} u^2 \right) dx \right\} ;$$

and this completes the proof of Theorem 4.3.

//

4.2 Boundary upper bounds and complete sharp description of the heat kernel for large values of time

Here we will first prove the following:

Theorem 4.4 *For $N \geq 2$, let $\Omega \subset \mathbf{R}^N$ be a smooth bounded and convex domain. Then there exists a positive constant C such that*

$$h(t,x,y) \leq C \frac{d^{\frac{1}{2}}(x) d^{\frac{1}{2}}(y)}{t^{\frac{1}{2}}} t^{-\frac{N}{2}}, \quad \forall\, x, y \in \Omega,\ t > 0 .$$

To this end we need the following estimate of [FMT2]:

Proposition 4.5 *For $N \geq 2$, let $\Omega \subset \mathbf{R}^N$ be a smooth bounded and convex domain. Then there exists a positive constant C such that*

$$\int_\Omega \left(|\nabla u|^2 - \frac{1}{4d^2(x)} u^2 \right) dx \geq C \left(\int_\Omega d^{\frac{q}{2}(N-2)-N}(x) |u|^q \, dx \right)^{\frac{2}{q}} , \tag{4.1}$$

for any $u \in C_0^\infty(\Omega)$ and any $2 < q \leq \frac{2N}{N-2}$ if $N \geq 3$ or any $2 < q < \infty$ if $N = 2$.

Using (4.1) the following logarithmic Sobolev inequality can be easily obtained

$$\int_\Omega v^2 \log\left(\frac{v}{||v||_2} \right) d \, dx \leq \epsilon \int_\Omega \left(|\nabla v|^2 d - \frac{1}{2} \Delta d \, v^2 \right) dx + \left(K_1 - \frac{N+1}{4} \log \epsilon \right) ||v||_2^2 , \tag{4.2}$$

for all $v \in C_0^\infty(\Omega)$, $v \geq 0$, and any $\epsilon > 0$; here K_1 is a positive constant independent of ϵ and $||v||_2 := \left(\int_\Omega |v|^2 \ d \ dx\right)^{\frac{1}{2}}$.

To obtain (4.2) we apply (4.1) to $v := ud^{-\frac{1}{2}}$ to get for any $v \in C_0^\infty(\Omega)$

$$\int_\Omega \left(|\nabla v|^2 d - \frac{1}{2}\Delta d \ v^2\right) dx \geq C \left(\int_\Omega v^q d^{\frac{q}{2}(N-2)-N+\frac{q}{2}} \ dx\right)^{\frac{2}{q}} .$$

Taking $q := \frac{2(N+1)}{N-1}$ we have

$$\int_\Omega \left(|\nabla v|^2 d - \frac{1}{2}\Delta d \ v^2\right) dx \geq C \left(\int_\Omega v^{\frac{2(N+1)}{(N-1)}} d \ dx\right)^{\frac{N-1}{N+1}} . \tag{4.3}$$

Then arguing in a quite similar way as in the proof of (3.7) in Subsection 3.1 we obtain (4.2) with $K_1 := \frac{N+1}{4} \log\left(\frac{N+1}{4C}\right)$.

Proof of Theorem 4.4: Let $H_0^1(\Omega, d \ dx)$ be the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$||v||_{H^1_{0,d}} := \left\{\int_\Omega \left(|\nabla v|^2 \ d + \frac{1}{2}(-\Delta d) v^2\right) dx\right\}^{\frac{1}{2}} .$$

Let $\bar{H} := U^{-1}HU$, $U : L^2(\Omega, d \ dx) \to L^2(\Omega)$ being the unitary operator $Uv := d^{\frac{1}{2}}v$, thus $\bar{H} := -\frac{1}{d}div(d\nabla) - \frac{1}{2}\frac{\Delta d}{d}$. To the operator $\bar{H}$ defined in the domain $D(\bar{H}) = \{v \in H_0^1(\Omega, d \ dx) : \bar{H}v \in L^2(\Omega, d \ dx)\}$ it is naturally associated the bilinear symmetric form defined as follows $\bar{\mathcal{H}}[v_1, v_2] :=< \bar{H}v_1, v_2 >_{L^2(\Omega, d \ dx)} =< v_1, v_2 >_{H_0^1(\Omega, d \ dx)}$ which is a Dirichlet form. Whence Lemma 1.3.4 and Theorems 1.3.2 and 1.3.3 in [D4] implies that $e^{-\bar{H}t}$, which is an analytic contraction semigroup in $L^2(\Omega, d \ dx)$, is also positivity preserving and a contraction semigroup in $L^p(\Omega, d \ dx)$ for any $1 \leq p \leq \infty$. As a consequence for any $t > 0$ and any $p \geq 2$

$$e^{-\bar{H}t}[L^2(\Omega, d \ dx) \cap L^\infty(\Omega)]^+ \subset [H_0^1(\Omega, d \ dx) \cap L^p(\Omega, d \ dx) \cap L^\infty(\Omega)]^+ ;$$

thus by density argument the L^p logarithmic Sobolev inequality, which can be deduced as usual from the L^2 logarithmic Sobolev inequality (4.2) (see Subsection 3.1 where a similar argument is used) more generally applies to any function in $\cup_{t>0} \ e^{-\bar{H}t}[L^2(\Omega, d \ dx) \cap L^\infty(\Omega)]^+$. This means that Theorem 2.2.7 in [D4] can be applied, as in Corollary 2.2.8 in [D4], to the operator $\bar{H}$; whence obtaining that

$$||e^{-\bar{H}t}||_{2\to\infty} \leq Ct^{-\frac{N+1}{4}} ,$$

and by duality that

$$||e^{-\bar{H}t}||_{1\to 2} \leq Ct^{-\frac{N+1}{4}} ,$$

that is

$$||e^{-\bar{H}t}||_{1\to\infty} \leq Ct^{-\frac{N+1}{2}} .$$

Here we use the following notation:

$$||e^{-\bar{H}t}||_{q\to p} := \sup_{0<||f||_q \leq 1} \frac{||e^{-\bar{H}t} f(x)||_p}{||f(x)||_q} ,$$

where $||f||_q := \left(\int_\Omega |f|^q \ d \ dx\right)^{\frac{1}{q}}$. This implies, by Dunford-Pettis theorem, that the semigroup $e^{-\bar{H}t}$ is indeed a semigroup of integral operators; that is a heat kernel $\bar{h}(t, x, y)$ associated to the semigroup $e^{-\bar{H}t}$

is well defined and satisfies the following pointwise upper bound $\bar{h}(t,x,y) \le C\frac{1}{t^{\frac{1}{2}}}t^{-\frac{N}{2}}$, for any $x, y \in \Omega$ and any $t > 0$. Theorem 4.4 then follows, due to the fact that, as a consequence of the unitary operator U, the heat kernels $h(t,x,y)$ and $\bar{h}(t,x,y)$, corresponding respectively to H and $\bar{H}$, satisfy the following equivalence $h(t,x,y) \equiv d^{\frac{1}{2}}(x)d^{\frac{1}{2}}(y)\ \bar{h}(t,x,y)$.

Remark 4.6 *Applying Davies's method of exponential perturbation to the operator $\bar{H}$ (see Section 2 in [D3] for details) the upper bound in Theorem 4.4 can be improved by adding a factor $c_\delta e^{-\frac{|x-y|^2}{4(1+\delta)t}}$.*

//

Let us now give the sketch of the proof of Theorem 1.4.

Proof of Theorem 1.4: Let us first improve by an exponential decreasing in time factor the global in time upper bound stated in Theorem 4.4. To this end let us define, as in Section 2 of [D2], the operator $\tilde{H} := U^{-1}(H - \lambda_1)U$, $U : L^2(\Omega, \varphi_1^2 dx) \to L^2(\Omega)$ being the unitary operator $Uw := \varphi_1 w$, thus $\tilde{H} := -\frac{1}{\varphi_1^2} div(\varphi_1^2 \nabla)$. Here $\varphi_1 > 0$ denotes the first eigenfunction and $\lambda_1 > 0$ the first eigenvalue corresponding to the Dirichlet problem $-\Delta\varphi_1 - \frac{1}{4d^2(x)}\varphi_1 = \lambda_1\varphi_1$ in Ω, $\varphi_1 = 0$ on $\partial\Omega$, normalized in such a way that $\int_\Omega \varphi_1^2(x)\ dx = 1$. Since it is known that there exist two positive constants c_1, c_2 such that

$$c_1\ d^{\frac{1}{2}}(x) \le \varphi_1(x) \le c_2\ d^{\frac{1}{2}}(x),\ \forall\ x \in \Omega \tag{4.4}$$

(as a consequence of Lemma 7 in [DD]), logarithmic Sobolev inequalities analogous to (4.2) also hold true if we replace $\bar{H}$ by $\tilde{H}$. Thus as a consequence of Gross theorem, exactly as in the proof of Theorem 4.4, the corresponding heat kernel $\tilde{h}(t,x,y)$ satisfies the same pointwise upper bound as $\bar{h}(t,x,y)$ that is $\tilde{h}(t,x,y) \le \frac{C}{t^{\frac{1}{2}}}t^{-\frac{N}{2}}$ for any $x, y \in \Omega$ and any $t > 0$. From the definition of U, it follows

$$h(t,x,y) \equiv \varphi_1(x)\varphi_1(y)\tilde{h}(t,x,y)e^{-\lambda_1 t}\ , \tag{4.5}$$

thus we get that $h(t,x,y) \le C\frac{d^{\frac{1}{2}}(x)d^{\frac{1}{2}}(y)}{t^{\frac{1}{2}}}t^{-\frac{N}{2}}e^{-\lambda_1 t}$ for any $x, y \in \Omega$ and any $t > 0$. Finally arguing as in Theorem 6 of [D1], an analogous lower estimate can be easily deduced (we refer to the proof of Theorem 1.2 where a similar argument is used).

//

4.3 Complete sharp description of the heat kernel for small values of time

In this section we prove the two-sided sharp estimate on the heat kernel $h(t,x,y)$ stated for small time in Theorem 1.3.

Before doing so let observe that Theorem 1.5 entails also the following parabolic Harnack inequality for the Schrödinger operator having critical singularity at the boundary

Theorem 4.7 *For $N \ge 2$, let $\Omega \subset \mathbf{R}^N$ be a smooth bounded and convex domain. Then there exist positive constants C_H and $R = R(\Omega)$ such that for $x \in \Omega$, $0 < r < R$ and for any positive solution $u(y,t)$ of $\frac{\partial u}{\partial t} = \Delta u + \frac{1}{4d^2(y)}u$ in $\{\mathcal{B}(x,r) \cap \Omega\} \times (0, r^2)$ we have the estimate*

$$ess\ sup_{(y,t)\in\left\{\mathcal{B}(x,\frac{r}{2})\cap\Omega\right\}\times(\frac{r^2}{4},\frac{r^2}{2})}\ u(y,t)d^{-\frac{1}{2}}(y) \le C_H\ ess\ inf_{(y,t)\in\left\{\mathcal{B}(x,\frac{r}{2})\cap\Omega\right\}\times(\frac{3}{4}r^2,r^2)}\ u(y,t)d^{-\frac{1}{2}}(y)\ .$$

Proof: As a first step let us observe that if u satisfies $u_t = -Hu$ then $v(y,t) := e^{\lambda_1 t}\varphi_1(y)^{-1}u(y,t)$ satisfies $v_t = -\tilde{H}v$. Whence as a consequence of (4.4), due to Remark 2.16, v satisfies Theorem 1.5 for $\alpha = 1$. From this the claim can be easily deduced.

//

The proof of Corollary 1.7 is similar to the above proof of Theorem 4.7, thus we omit the details.

We are now ready to prove Theorem 1.3.

Proof of Theorem 1.3 Since for any $x \in \Omega$ and for some positive constants c_1, c_2 we have the following estimate $c_1 d^{\frac{\alpha}{2}}(x) \leq \varphi_1(x) \leq c_2 d^{\frac{\alpha}{2}}(x)$ for $\alpha = 1$, we can apply the result of Theorem 2.15 to the operator $\tilde{H} = -\frac{1}{\varphi_1^2(x)} div(\varphi_1^2(x)\nabla)$. Hence due to (4.5) the result follows immediately.

//

The proof of Corollary 1.8 is similar to the above proofs of Theorems 1.3 and 1.4, thus we omit the details.

Let us finally make some remarks concerning Schrödinger operators having potential $V(x) = cd^{-2}(x)$. Arguing as in Lemma 7 in [DD] one can easily prove that the first Dirichlet eigenfunction for the Schrödinger operator $-\Delta - \frac{c}{d^2(x)}$, $0 < c < \frac{1}{4}$, behaves like $d^{\frac{\alpha}{2}}$ on all Ω, for $\alpha := 1 + \sqrt{1-4c}$. Then we have:

Theorem 4.8 *For $N \geq 2$, let $\Omega \subset \mathbf{R}^N$ be a smooth bounded and convex domain. Then there exist positive constants C_1, C_2, with $C_1 \leq C_2$, and $T > 0$ depending on Ω such that*

$$C_1 \min\left\{1, \frac{d^{\frac{\alpha}{2}}(x) d^{\frac{\alpha}{2}}(y)}{t^{\frac{\alpha}{2}}}\right\} t^{-\frac{N}{2}} e^{-C_2 \frac{|x-y|^2}{t}} \leq h_c(t,x,y) \leq C_2 \min\left\{1, \frac{d^{\frac{\alpha}{2}}(x) d^{\frac{\alpha}{2}}(y)}{t^{\frac{\alpha}{2}}}\right\} t^{-\frac{N}{2}} e^{-C_1 \frac{|x-y|^2}{t}}$$

for all $x, y \in \Omega$ and $0 < t \leq T$; where $h_c(t,x,y)$ denotes the heat kernel associated to the operator $-\Delta - \frac{c}{d^2(x)}$ in Ω under Dirichlet boundary conditions, for any $0 < c < \frac{1}{4}$ and $\alpha := 1 + \sqrt{1-4c}$.

Theorem 4.9 *For $N \geq 2$, let $\Omega \subset \mathbf{R}^N$ be a smooth bounded and convex domain. Then there exist two positive constants C_1, C_2, with $C_1 \leq C_2$, such that*

$$C_1 \; d^{\frac{\alpha}{2}}(x) \; d^{\frac{\alpha}{2}}(y) \; e^{-\lambda_1 t} \leq h_c(t,x,y) \leq C_2 \; d^{\frac{\alpha}{2}}(x) \; d^{\frac{\alpha}{2}}(y) \; e^{-\lambda_1 t}$$

for all $x, y \in \Omega$ and $t > 0$ large enough; where $h_c(t,x,y)$ denotes the heat kernel associated to the operator $-\Delta - \frac{c}{d^2(x)}$ in Ω under Dirichlet boundary conditions, λ_1 its (positive) elliptic first eigenvalue, for any $0 < c < \frac{1}{4}$ and $\alpha := 1 + \sqrt{1-4c}$.

Remark 4.10 *Let us at this point remark that Theorems 1.3 and 4.8 as well as Theorem 4.7 concerning respectively sharp asymptotic for small time and the parabolic Harnack inequality up to the boundary for the Schrödinger operator having potential $V(x) = cd^{-2}(x)$, hold true also without any convexity assumption on the domain Ω under consideration.*

4.4 On Davies conjecture

In this subsection we consider Davies conjecture. For this we suppose that $\widetilde{E}$ denotes the self-adjoint operator associated with the closure of the positive quadratic form

$$Q(f) = \int_\Omega \left(\sum_{i,j=1}^N a_{i,j}(x) \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j} - V f^2 \right) dx \ ,$$

initially defined on $C_0^\infty(\Omega)$; where $(a_{i,j}(x))_{N\times N}$ is a measurable symmetric uniformly elliptic matrix such that

$$\sum_{i,j=1}^{N} a_{i,j}(x)\xi_i\xi_j \geq |\xi|^2$$

and V is a potential on Ω such that

$$V(x) = V_1(x) + V_2(x)\ , \tag{4.6}$$

where

$$|V_1(x)| \leq \frac{1}{4d^2(x)}\ , \quad V_2(x) \in L^p(\Omega),\ p > \frac{N}{2}\ . \tag{4.7}$$

We also suppose that

$$\lambda_1 := \inf_{0\neq\varphi\in C_0^\infty(\Omega)} \frac{Q(\varphi)}{\int_\Omega \varphi^2 dx} > 0, \tag{4.8}$$

and that to λ_1 there corresponds a positive eigenfunction φ_1 satisfying for all $x\in\Omega$ the following estimate,

$$c_1\ d^{\frac{\alpha}{2}}(x) \leq \varphi_1(x) \leq c_2\ d^{\frac{\alpha}{2}}(x)\ , \quad \text{for some} \quad \alpha \geq 1 \tag{4.9}$$

and for $c_1,\ c_2$ two positive constants.

Thus $\widetilde{E}$ is defined on the closure of $C_0^\infty(\Omega)$ with respect to the norm defined by the quadratic form Q. Then as before it can be shown that $\widetilde{E}$ is a well defined nonnegative self-adjoint operator on $L^2(\Omega)$ such that for every $t>0$, $e^{-\widetilde{E}t}$ has a integral kernel, that is $e^{-\widetilde{E}t}u_0(x) := \int_\Omega \widetilde{e}(t,x,y)u_0(y)dy$ and if $N\geq 3$ a Green function $G_{\widetilde{E}}(x,y) = \int_0^\infty \widetilde{e}(t,x,y)dt$ denoting the kernel of $\widetilde{E}^{-1}$.

Theorem 4.11 *For $N\geq 3$, let $\Omega\subset \mathbf{R}^N$ be a smooth bounded domain. Suppose that (4.6), (4.7), (4.8) and (4.9) are satisfied. Then there exist two positive constants C_1, C_2, with $C_1\leq C_2$, such that for any $x,y\in\Omega$*

$$C_1 \min\left\{\frac{1}{|x-y|^{N-2}}, \frac{d^{\frac{\alpha}{2}}(x)d^{\frac{\alpha}{2}}(y)}{|x-y|^{N+\alpha-2}}\right\} \leq G_{\widetilde{E}}(x,y) \leq C_2 \min\left\{\frac{1}{|x-y|^{N-2}}, \frac{d^{\frac{\alpha}{2}}(x)d^{\frac{\alpha}{2}}(y)}{|x-y|^{N+\alpha-2}}\right\}.$$

Davies conjecture is stated under slightly stronger assumptions on V than (4.7) and on φ_1 (only when $\alpha=2$).

Proof: We note that we have $E_1 := -\frac{1}{\varphi_1^2}div(\varphi_1^2\nabla) \equiv U^{-1}(\widetilde{E}-\lambda_1)U$, $U: L^2(\Omega,\varphi_1^2dx)\to L^2(\Omega)$ being the unitary operator $Uw := \varphi_1 w$; hence we have the following relationship between heat kernels

$$\widetilde{e}(t,x,y) = \varphi_1(x)\varphi_1(y)e_1(t,x,y)e^{-\lambda_1 t}\ . \tag{4.10}$$

Due to (4.9) we can apply Theorem 2.15 to the operator E_1. Hence due to (4.10) for two positive constants $C_1\leq C_2$, we have for small time

$$C_1 \min\left\{1, \frac{d^{\frac{\alpha}{2}}(x)d^{\frac{\alpha}{2}}(y)}{t^{\frac{\alpha}{2}}}\right\} t^{-\frac{N}{2}} e^{-C_2\frac{|x-y|^2}{t}} \leq \widetilde{e}(t,x,y) \leq C_2 \min\left\{1, \frac{d^{\frac{\alpha}{2}}(x)d^{\frac{\alpha}{2}}(y)}{t^{\frac{\alpha}{2}}}\right\} t^{-\frac{N}{2}} e^{-C_1\frac{|x-y|^2}{t}}. \tag{4.11}$$

On the other hand for large time

$$C_1\ d^{\frac{\alpha}{2}}(x)\ d^{\frac{\alpha}{2}}(y)\ e^{-\lambda_1 t} \leq \widetilde{e}(t,x,y) \leq C_2\ d^{\frac{\alpha}{2}}(x)\ d^{\frac{\alpha}{2}}(y)\ e^{-\lambda_1 t}, \tag{4.12}$$

for all $x,y\in\Omega$. To obtain this estimate we need to prove a global Sobolev inequality on Ω, which can be easily deduced from its local version (2.12) as well as (2.18) with $\lambda=0$ there, by means of a partition of unity as in [K]. Then the result follows integrating $\widetilde{e}(t,x,y)$ in the time variable.

References

[A] Aronson D.G. Bounds for the fundamental solution of a parabolic equation, Bull. Amer. Math. Soc. **73**, (1967), 890-896.

[BG] Baras P. and Goldstein J. The heat equation with a singular potential, Trans. Amer. Math. Soc. **284**, (1984), 121-139.

[BFT1] Barbatis G., Filippas S. and Tertikas A. A unified approach to improved L^p Hardy inequalities with best constants, Trans. Amer. Math. Soc. **356** n. 6 (2003), 2169-2196.

[BFT2] Barbatis G., Filippas S. and Tertikas A. Critical heat kernel estimates for Schrödinger operators via Hardy-Sobolev inequalities, J. Funct. Anal. **208** (2004), 1-30.

[BM] Brezis H. and Marcus M. Hardy's inequalities revised. Dedicated to Ennio De Giorgi. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 25 (1997), n. 1-2, 217-237.

[BV] Brezis H. and Vazquez J.L. Bolw-up solutions of some nonlinear elliptic problems, Rev. Mat. Univ. Complut. Madrid 10 n. 2 (1997), 443-469.

[CM] Cabré X. and Martel Y. Existence versus explosion instantanée pour des équationes de la chaleur linéaires avec potentiel singulier, C.R. Acad. Sci. Paris Ser. I Math. **329**, n. 11 (1999), 973-978.

[CS] Chiarenza F.M. and Serapioni R.P. A remark on a Harnack inequality for degenerate parabolic equations, Rend. Sem. Mat. Univ. Padova **73**, (1985), 179-190.

[D1] Davies E.B. Perturbations of ultracontractive semigroups, Quart. J. Math. Oxford **2**, 37 (1986), 167-176.

[D2] Davies E.B. The equivalence of certain heat kernel and Green function bounds, J. Funct. Anal. **71** (1987), 88-103.

[D3] Davies E.B. Explicit constants for Gaussian upper bounds on heat kernels, Amer. J. of Math. **109**, (1987), 319-334.

[D4] Davies E.B. Heat kernels and spectral theory, Cambridge University Press (1989).

[DS1] Davies E.B. and Simon B. Ultracontractivity and the heat kernels for Schrodinger operators and Dirichlet Laplacians, J. Funct. Anal. **59** (1984), 335-395.

[DS2] Davies E.B. and Simon B. L^p norms of non-critical Schrödinger semigroups, J. Funct. Anal. **102** (1991), 95-115.

[DD] Dávila J. and Dupaigne L. Hardy type inequalities, J. Eur. Math. Soc. **6** n. 3 (2004), 335–365.

[DG] De Giorgi E. Sulla differenziabilitá e l'analiticitá delle estremali degli integrali multipli regolari. Mem. Accad. Sci. Torino Cl. Sci. Fis. Mat. Nat. (3) n. 3 (1957), 25-43.

[FGS] Fabes E.B., Garofalo N. and Salsa S. A backward Harnack inequality and Fatou Theorem for nonnegative solutions of parabolic equations, Illinois J. of Math. **30**, n. 4 (1986), 536-565.

[FKJ] Fabes E. B., Kenig C. E. and Jerison D. Boundary behavior of solutions to degenerate elliptic equations. Conference on harmonic analysis in honor of Antoni Zygmund, Vol. I, II (Chicago, Ill., 1981), 577–589, Wadsworth Math. Ser., Wadsworth, Belmont, CA, 1983.

[FKS] Fabes E.B., Kenig C. E. and Serapioni R.P. The local regularity of solutions of degenerate elliptic equations, Comm. Part. Diff. Eq. **7**, (1982), 77-116.

[FS] Fabes E.B. and Stroock D.W. A new proof of Moser's parabolic Harnack inequality via the old ideas of Nash, Arch. Rat. Mech. Anal. **96**, (1986), 327-338.

[FMT1] Filippas S., Maz'ya V. G. and Tertikas A. Sharp Hardy-Sobolev inequalities, C.R. Math. Acad. Sci. Paris **339** (2004), n. 7, 483-486.

[FMT2] Filippas S., Maz'ya V. G. and Tertikas A. Critical Hardy-Sobolev inequalities, in preparation.

[FT] Filippas S. and Tertikas A. Optimizing Improved Hardy inequalities, J. Funct. Anal. **192** (2002), 186-233.

[Ga] Gao P. The boundary Harnack principle for some degenerate elliptic operators, Comm. Partial Differential Equations **18** (1993) n. 12, 2001-2022.

[G1] Grigoryan A. The heat equation on non-compact Riemannian manifolds, (in Russian) Matem. Sbornik **182** (1991) n. 1, 55-87. Engl. transl. : Math. USSR Sb. **72**(1992) n. 1, 47-77.

[G2] Grigoryan A. Heat kernels on weighted manifolds and applications, to appear in Cont. Math. (2006), 91-190.

[GSC] Grigoryan A. and Saloff-Coste L. Stability results for Harnack inequalities, Ann. Inst. Fourier, Grenoble **55**, n. 3 (2005), 825-890.

[K] Kufner A. Weighted Sobolev spaces, Teubner-Texte zur Mathematik, 31 (1981).

[KO] Kufner A. and Opic B. Hardy type inequalities, Pitman Research Notes in Math Series **219** (1990)

[LY] Li P. and Yau S.-T. On the parabolic kernel of the Schrödinger operator, Acta Math. **156**, n. 3-4 (1986), 153-201.

[M] Mazya V. G. Sobolev spaces, Springer Verlag (1985).

[MS] Milman P.D. and Semenov Yu.A. Global heat kernel bounds via desingularizing weights, J. Funct. Anal. **212** (2004), 373-398.

[MT1] Moschini L. and Tesei A. Harnack inequality and heat kernel estimates for the Schrödinger operator with Hardy potential, to appear in Rendiconti dell'Accademia Nazionale dei Lincei.

[MT2] Moschini L. and Tesei A. Parabolic Harnack inequality for the heat equation with inverse-square potential, to appear in Forum Mathematicum.

[Mo1] Moser J. On Harnack's theorem for elliptic differential equations, Comm. Pure Appl. Math. **14** (1961), 577-591.

[Mo2] Moser J. A Harnack inequality for parabolic differential equations, Comm. Pure. Appl. Math. **17** (1964), 101-134; Correction: **20** (1967), 231-236.

[N] Nash J. Continuity of solutions of parabolic and elliptic equations, Amer. J. Math. **80** (1958), 931-954.

[SC1] Saloff-Coste L. A note on Poincaré, Sobolev, and Harnack inequalities, Internat. Math. Res. Notices **2** (1992) 27-38.

[SC2] Saloff-Coste L. Aspects of Sobolev-type inequalities, London Math. Soc. Lecture Notes Series **289**, Cambridge University Press (2002).

[VZ] Vázquez J.L. and Zuazua E. The Hardy inequality and the asymptotic behaviour of the heat equation with an inverse-square potential, J. Funct. Anal. **173** (2000), 103-153.

[Z] Zhang, Qi. S. The boundary behaviour of heat kernels of Dirichlet Laplacians, J. Diff. Eq. **182** (2002), 416-430.